AD-A099 348 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
UPWIND DIFFERENCE SCHEMES FOR SYSTEMS OF CONSERVATION LAWS - PO--ETC(U)
MAR 81 B ENGQUIST, S OSHER DAAG29-80-C-0041
UNCLASSIFIED MRC-TSR-2186 NL

1 OF 1
AD A
099348

END
DATE
FILMED
6-81
DTIC

7 F (9)

MRC Technical Summary Rep... #2186 SC

(6) UPWIND DIFFERENCE SCHEMES FOR SYSTEMS OF CONSERVATION LAWS - POTENTIAL FLOW EQUATIONS,

(10) Bjorn Engquist [redacted] Stanley Osher

AD A099348

(12) 50

(14) MRC-TSR-2186

(15) DAAG29-80-C-0041

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

(11) Mar[redacted]981

Received September 8, 1980

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

NASA-Ames University Consortium
Los Angeles, California 90024

JCB

221200 81 5 27 018

UNIVERSITY OF WISCONSIN
MATHEMATICS RESEARCH CENTER

UPWIND DIFFERENCE SCHEMES FOR SYSTEMS OF CONSERVATION LAWS - POTENTIAL FLOW EQUATIONS

Bjorn Engquist† and Stanley Osher††

TECHNICAL SUMMARY REPORT # 2186
March 1981

ABSTRACT

We derive new upwind type finite difference approximations to systems of nonlinear hyperbolic conservation laws. The general technique is exemplified by the potential flow equations written as a first order system. The scheme has desirable properties for shock calculations. For the potential flow approximation, we show that the entropy condition is valid for limit solutions and that there exist discrete steady shocks which are unique and sharp. Numerical examples are given.

AMS(MOS) Subject Classification: 65M05

Key Words: Finite difference approximations, upwind schemes, potential flow equations, entropy condition, discrete shocks.

Work Unit Number 3 - Numerical Analysis and Computer Science

---

†This paper was written while the author was visiting the Mathematics Research Center at the University of Wisconsin.

†,††Address of the Authors: Department of Mathematics, University of California, Los Angeles, CA 90024.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and by NASA-Ames University Consortium, Interchange No. NCA2-OR390-002.

SIGNIFICANCE AND EXPLANATION

Hyperbolic systems of conservation laws are often used to describe compressible fluid flow. These hyperbolic partial differential equations can be approximated by finite difference schemes which in turn can be coded for computer calculations of practical problems. Aerodynamics is a typical field of application.

Standard difference schemes often run into difficulties when the solution to be approximated contains discontinuities in the form of shocks and contact discontinuities. The computed solution will typically either be smeared i.e. too smooth or will contain unphysical overshoots and wiggles.

For a large class of scalar problems it has been possible to design difference schemes of upwind type which produce approximations of solutions with shocks which are very sharp and without overshoots. In an upwind scheme all differences are one sided and the structure usually depends on the solution itself.

This paper describes a systematic way of deriving difference schemes of upwind type for a class of hyperbolic systems of conservation laws. Many of the desirable properties which upwind schemes have for scalar problems can thus be extended to the physically much more important case of systems.

This technique is used to produce a scheme for the potential flow equations. It is proved that only physically permitted shocks will be approximated in the limit and that steady shocks are very sharp. Other cases are investigated in computational examples which also display the efficiency of the scheme.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC and not with the authors of this report.

UPWIND DIFFERENCE SCHEMES FOR SYSTEMS OF CONSERVATION LAWS - POTENTIAL FLOW EQUATIONS

Bjorn Engquist† and Stanley Osher††

1. INTRODUCTION

Many upwind difference schemes have very attractive properties when approximating scalar nonlinear hyperbolic conservation laws. They have, in particular, become the standard technique for many calculations of transonic flow [1], [3], [7], [10], [12], [16]. There are, for example, several versions of upwind schemes for the approximation of the small disturbance equation of transonic flow (1.1). A number of these schemes have solutions with sharp shock profiles [1], [5], [7], [16]. The small disturbance equation is

$$2\varphi_{tx} = \left(K\varphi_x - \tfrac{1}{2}(\gamma+1)\varphi_x^2\right)_x + \varphi_{yy} \ . \tag{1.1}$$

The velocity potential is denoted by $\varphi(x,y,t)$ and K and γ are positive constants.

The extension of these techniques to systems is immediate when all the eigenvalues of the Jacobian matrix of the flux functions have the same sign. It is the purpose of this paper to present a systematic technique for producing upwind difference schemes for the more interesting case of systems where the eigenvalues of the Jacobian may have different signs. We shall, as an example, apply this technique to the potential flow equations (1.2) for compressible, inviscid, isentropic and irrotational flow [3]. The equation is

$$\rho_t + (\rho\varphi_x)_x + (\rho\varphi_y)_y = 0 \ . \tag{1.2}$$

The density function ρ is given in terms of a velocity potential φ through Bernoulli's law

†This paper was written while the author was visiting the Mathematics Research Center at the University of Wisconsin.

†,††Address of the authors: Department of Mathematics, University of California, Los Angeles, Ca. 90024.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and by NASA-Ames University Consortium, Interchange No. NCA2-OR390-002.

$$(1.3)\qquad \rho^{\gamma-1} + \frac{\gamma-1}{2A\gamma}(2\varphi_t + \varphi_x^2 + \varphi_y^2) = H(t) .$$

The equation of state for the pressure p is

$$(1.4)\qquad p = A\rho^{\gamma}$$

with A abd γ positive constants ($1 < \gamma < 3$ in our theorems). We can transform (1.2), (1.3) into a first order hyperbolic system by letting $\varphi_x = u$, $\varphi_y = v$ and then differentiating (1.3) with respect to x and y resepectively [2]. The equality of the mixed partials is assumed to be valid throughout the flow. the system of equations is

$$(1.5)\qquad \begin{pmatrix} \rho \\ u \\ v \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \frac{1}{2}(u^2+v^2) + \frac{A\gamma}{\gamma-1}\rho^{\gamma-1} \\ 0 \end{pmatrix}_x + \begin{pmatrix} \rho v \\ 0 \\ \frac{1}{2}(u^2+v^2) + \frac{A\gamma}{\gamma-1}\rho^{\gamma-1} \end{pmatrix}_y = 0 .$$

Analogous to a standard procedure for scalar problems one might use dimensional splitting for the solution of equations with more than one space variable. In this paper we shall consider the reduced one dimensional system

$$(1.6)\qquad \begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \frac{1}{2}u^2 + \frac{A\gamma}{\gamma-1}\rho^{\gamma-1} \end{pmatrix}_x = 0 .$$

Let us recall the first order scalar upwind scheme which we developed in [5], [7]. Consider a nonlinear scalar conservation law (1.7) in one space dimension

$$(1.7)\qquad u_t + f(u)_x = 0 , \quad t > 0, \quad -\infty < x < \infty$$

$$(1.8)\qquad u(x,0) = u(x).$$

The solution $u(x,t)$ is approximated by a mesh function u_j^n on the mesh $\{(x_j,t^n)\}$, $x_j = j\Delta x$, $t^n = n\Delta t$, ($u_j^n \approx u(x_j,t^n)$). The difference scheme in its explicit form is

$$(1.9)\qquad u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(\Delta_+ f_-(u_j^n) + \Delta_- f_+(u_j^n))$$

$$(1.10)\qquad u_j^0 = u(x_j,0).$$

The difference operators Δ_+ and Δ_- denote the forward and backward differencing respectively $(\Delta_\pm u_j = \pm(u_{j\pm1} - u_j))$. The auxiliary function f_- and f_+ contains the increasing and decreasing parts of f respectively,

$$f_+(u) = \int_0^u \chi(s)f'(s)ds \tag{1.11}$$

$$f_-(u) = \int_0^u (1-\chi(s))f'(s)ds \tag{1.12}$$

$$f = f_+ + f_-$$

When f is convex the definitions (1.11) and (1.12) simplifies to

$$f_+(u) = f(\max(u,\bar{u})) \tag{1.13}$$

$$f_-(u) = f(\min(u,\bar{u})) \tag{1.14}$$

where $\bar{u}$ is the stagnation point $f'(\bar{u}) = 0$. If f' has a fixed sign say $f' > 0$ the scheme reduces to the classical upwind scheme

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\Delta_- f(u_j^n) \; . \tag{1.15}$$

When it is used as the basic ingredient in the approximation of the *small disturbance* equation (1.1) the resulting scheme will be identical to the Cole-Murman scheme [16] away from sonic and shock points. These are the points where f' changes sign and a switch from $\Delta_- f$ to $\Delta_+ f$ is needed. The formulation (1.9) gives a recipe for such a switch.

The approximation (1.9) has several attractive properties in connection with shock calculations. Proofs and numerical examples are given in [5], [6].

These properties are:

(a) The scheme is monotone, see [4], [9], for $\Delta t|f'| \le \Delta x$.

(b) It is in conservation form and hence produces shocks with the correct location [14].

These properties imply the following properties (c), (d) and (e) for Cauchy problems. (We thank P. Lax and M. Crandall for some helpful discussions on this matter.) In our earlier work we proved the results below for quadratic f even for the mixed initial boundary value problem, and for general scalar f for the Cauchy problem.

(c) An entropy condition is valid for limit solutions, which rules out nonphysical shocks.

(d) The scheme is stable in L_1, L_2 and L_∞.

(e) The approximate solution u_j^n converges to u in L_1.

(f) The approximation uses the same number of boundary conditions as the differential equation, i.e. *no* extra numerical boundary conditions need to be imposed.

(g) Shock solutions have stable sharp shock profiles, [5], [6]. We essentially mean that the approximation of a steady Riemann problem is exact two points away from the shock.

We now try to preserve some of these properties in the approximation of systems. We shall consider a hyperbolic system of nonlinear conservation laws in one space dimension

$$u_t + f(u)_x = 0\ , \quad u : R^2 \to R^d. \tag{1.16}$$

Systems in more than one space variable can be reduced to the one dimensional case by dimensional splitting or ADI, see [1], [5].

The linear stability requirement implies that there only exists strictly upwind difference schemes if all the eigenvalues of the Jacobian matrix ∂f have the same sign. These eigenvalues are all real since the system (1.16) is hyperbolic. This stability requirement follows directly from the domain of dependence and the CFL-condition. The upwind difference scheme may, for example, be of the simple form (1.15) if all eigenvalues of ∂f are positive.

We have to clarify what we mean by upwind or one sided difference schemes when ∂f has eigenvalues of different signs in some region of a solution space. Difference methods for which the approximation of the spatial derivatives are non symmetric and may change when the signs of the eigenvalues of ∂f changes are often said to be of upwind type. We shall consider here a special class of such schemes for which it is possible to prove that the properties (b), (c) and (g) are valid when approximating the system of equations (1.6). The property (f) is valid in a slightly weaker form. Other upwind schemes for systems are presented in [2], [18], [19].

The sharp shock profile property (g) is of computational importance and forces the scheme to be of a special structure. It is otherwise easy to produce a linearly stable scheme of type (1.9) which approximates (1.16) and is in conservation form. Choose $f_+ = \frac{1}{2}(f + cu)$ and $f_- = \frac{1}{2}(f - cu)$ where c is a constant such that $c > \rho(\partial f)$. The scheme is linearly stable for $\Delta t/\Delta x$ small enough. However, not even for scalar problems, $(d = 1)$, with steady shocks will the numerical solution have a sharp profile. This follows from [11] since this scheme is strictly monotone. (If $u_j^{n+1} = G(u_{j+1}^n, u_j^n, u_{j-1}^n)$ then $\frac{\partial G}{\partial u_k} > 0$ for $k = j+1, j, j+1$).

The strictly upwind scalar scheme (1.9), (1.11), (1.12) is monotone but not strictly monotone since at most one of f'_+ and f'_- is non zero for each u. The natural generalization to systems is

$$N(\partial f_+(u)) + N(\partial f_-(u)) \leq d \tag{1.17}$$

where N is the counting function for the number of nonzero eigenvalues. This property is also basic for (f) to be valid. We also need $f = f_+ + f_-$ (modulo a constant) and the matrices ∂f_+ and $-\partial f_-$ must have nonnegative eigenvalues for linear stability.

When the definition (1.11) and (1.12) of f_+ and f_- respectively are used in (1.9) the scheme can be written

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(\int_{u_j^n}^{u_{j+1}^n} (1-\chi(s))f'(s)ds + \int_{u_{j-1}^n}^{u_j^n} \chi(s)f'(s)ds \right. . \tag{1.18}$$

This is the form of the algorithm that will be used for systems. In Section 2 we shall present the algorithm for choosing the matrix $\chi(s)$ and the path of integration between u_j^n and u_{j+1}^n. The choice of the path of integration is based on the Riemann invariant curves and it is crucial for the success of this scheme.

In Section 3 we shall derive the explicit form of the difference approximation for the one dimensional full potential equation (1.6). We shall also remark about boundary conditions (f) and show that (1.17) is true for constant states.

We shall prove that the entropy condition is valid for limit solutions of the approximation in Section 4. The global existence and uniqueness of discrete, steady one and two shocks is also proved. These shocks are equal to the analytic shocks two mesh points away from the discontinuity.

In Section 5 we shall present results from numerical calculation with the scheme derived in Section 3.

2. THE GENERAL ALGORITHM

In this section we shall present the derivation of the upwind algorithm for systems in some generality. See also [17] for a related discussion of this general technique.

Consider a strictly hyperbolic system of nonlinear conservation laws in one space dimension

$$u_t + f(u)_x = 0 \ , \quad u: R^2 \to R^d, \quad t > 0 \ , \quad -\infty < x < \infty \tag{2.1}$$

$$u(x,0) = u(x). \tag{2.2}$$

The flux vector f is assumed to have continuous derivatives. The difference approximation is defined as follows

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} \left(\int_{u_j^n}^{u_{j+1}^n} (1 - \chi(u))\partial f(u)\,du + \int_{u_{j-1}^n}^{u_j^n} \chi(u)\partial f(u)\,du \right) \tag{2.3}$$

$$u_j^0 = u(x_j) \ . \tag{2.4}$$

The matrix $\chi(s)$ and the paths of integration remain to be specified. Extending the principles from the scalar appoximation, the matrix χ should be the natural projection on R^d onto the subspace spanned by the eigenvectors corresponding to the positive eigenvalues of the Jacobian matrix ∂f. These eigenvectors are linearly independent since the system is strictly hyperbolic.

Let $T(u)$ be the matrix the columns of which are the eigenvectors of ∂f

$$T^{-1}(u)\ \partial f(u)\ T(u) = \Lambda(u) \tag{2.5}$$

$$\Lambda(u) = \mathrm{diag}\{\lambda_k(u)\} = \begin{pmatrix} \lambda_1(u) & 0 & \cdots & 0 \\ 0 & \lambda_1(u) & & \vdots \\ \vdots & & & \\ 0 & \cdots & & \lambda_d(u) \end{pmatrix} \tag{2.6}$$

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_d(u) \tag{2.7}$$

$$\chi(u) = T(u)\ \mathrm{diag}\{ \tfrac{1}{2} + \tfrac{1}{2}\,\mathrm{sign}[\lambda_k(u)]\} T^{-1}(u) \tag{2.8}$$

$$\chi(u)\partial f(u) = T(u)\mathrm{diag}\{\max(\lambda_k(u),0)\}T^{-1}(u) = (\partial f(u))^+ \tag{2.9}$$

$$(I - \chi(u))\partial f(u) = T(u)\mathrm{diag}\{\min(\lambda_k(u),0)\}T^{-1}(u) = (\partial f(u))^- . \tag{2.10}$$

The choice of path of integration significally affects properties of the scheme. The path should be specified in order to simplify the computations and to guarantee good characteristics of the solution. The definition will be connected to classical techniques for solving Riemann problems, [13], but will be much simpler.

Denote the path connecting u^n_{j-1} to u^n_j by Γ^j (and of course Γ^{j+1} connects u^n_j to u^n_{j+1}). The n dependence in Γ is omitted in the notation. The curve Γ^j is decomposed into d subcurves $\{\Gamma^j_k\}^d_{k=1}$

$$\Gamma^j = \bigcup_{k=1}^{d} \Gamma^j_k . \tag{2.11}$$

These subcurves are related to rarefaction solutions and are defined through

$$\Gamma^j_k : \begin{cases} \dfrac{du^{(k)}}{ds} = r_k(u^{(k)}) & 0 < s < s_k \quad \text{or} \quad s_k < s < 0 \\[2ex] u^{(k)}(0) = u^{(k)} & k = 1,\cdots,d \end{cases} \tag{2.12}$$

where $r_k(u)$ are the right normalized eigenvectors of $\partial f(u)$ corresponding to the eigenvalues $\lambda_k(u)$. The curves are connected by continuity conditions

$$\begin{cases} u^{(d)} = u^n_{j-1} \\ u^{(k-1)} = u^{(k)}(s_k) \quad , \qquad k = d,\cdots,2 . \\ u^{(1)}(s_1) = u^n_j \end{cases} \tag{2.13}$$

Note that the curve Γ^j starts at u^n_{j-1} with Γ^j_d corresponding to the eigenvector r_d. Then Γ^j continues with Γ^j_{d-1} etc.

The existence and uniqueness of a solution to (2.11), (2.12), which is the existence of Γ^j is guaranteed if u_j^n and u_{j-1}^n are not too far apart. This follows from the fact that the vectors $r_k(u)$, $k = 1,\cdots,d$ are linearly independent and that r_k depends continuously on u. In other words Γ_k^j, $k = 1,\cdots,d$ locally acts as a coordinate system for R^d.

An important property with this choice of path is that the system decouples in the following sense

$$(2.14)\quad \int_{\Gamma_k^j} \chi(u)\partial f(u)du = \int_{\Gamma_k^j} \chi(u(s))\partial f(u(s))r_k(u(s))ds = \int_0^{s_k^j} \max(\lambda_k(u_k(s)),0)r_k(u(s))ds$$

This follows from (2.9) and (2.12). The scheme (2.3) can hence be rewritten

$$(2.15)\quad u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\Big\{\sum_{k=1}^{d}\Big(\int_0^{s_k^j}\max(\lambda_k(u(s)),0)r_k(u(s))ds + \int_0^{s_k^{j+1}} min(\lambda_k(u(s)),0)r_k(u(s))ds\Big)\Big\}.$$

The eigenvectors r_k and the curves Γ_k have in many physically important problems a simple analytic form. This is e.g. the case for the full potential equation and the Euler equations.

We shall end this section by showing that the scheme (2.3) is in conservation form and is first order accurate.

$$u_j^{n+1} - u_j^n = -\frac{\Delta t}{\Delta x}\Big(\int_{\Gamma^{j+1}}(1-\chi(u))\partial f(u)du + \int_{\Gamma^j}\chi(u)\partial f(u)du\Big)$$

$$= -\frac{\Delta t}{\Delta x}\Big(\int_{\Gamma^{j+1}}(\partial f(u))^- du + \int_{\Gamma^j}(\partial f(u))^+ du\Big)$$

$$= -\frac{\Delta t}{\Delta x}\Big(\Delta_- f(u) + \Delta_+\int_{\Gamma^j}(\partial f(u))^+ du\Big)$$

$$= -\frac{\Delta t}{\Delta x}\Big(\Delta_+ f(u) - \Delta_+\int_{\Gamma^j}(\partial f(u))^- du\Big)$$

$$= -\frac{\Delta t}{\Delta x}\Big(\tfrac{1}{2}\Delta_0 f(u) + \tfrac{1}{2}\Delta_+\int_{\Gamma^j}|\partial f(u)|du\Big)$$

3. THE POTENTIAL FLOW ALGORITHM AND REMARKS ABOUT BOUNDARY CONDITIONS

We shall now derive the difference approximation for the one dimensional full potential equation

$$\rho_t + (\nabla\varphi_x)_x = 0 \tag{3.1}$$

where ρ is a known function of φ defined through Bernoulli's law

$$\varphi_t + \tfrac{1}{2}\varphi_x^2 + \frac{c^2}{\gamma-1} = H(t) \tag{3.2}$$

with $H(t)$ given and

$$c^2(\rho) = \frac{dp}{d\rho}\,. \tag{3.3}$$

The equation of state for the pressure is given to

$$p = A\rho^\gamma\ ,\quad A > 0\ ,\quad 1 < \gamma \leq 3\ . \tag{3.4}$$

In view of the general procedure outlined in the introduction and Section 2, we shall first treat the set of equations as a hyperbolic system for the two unknowns ρ and $u = \varphi_x$. After constructing this approximation, we shall in a future paper obtain a scalar difference scheme approximating (3.1) and (3.2).

By taking the space derivatives of (3.2) and using equality of the mixed partials for φ (which is assumed to hold even across discontinuities), we construct a first order strictly hyperbolic system of conservation laws. (See also (1.6) in the introduction.)

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \frac{u^2}{2} + \frac{c^2}{\gamma-1} \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.5}$$

The 2 by 2 Jacobian matrix

$$A = \begin{pmatrix} u & \rho \\ \frac{c^2}{\rho} & u \end{pmatrix} = \partial f \tag{3.6}$$

has distinct eigenvalues

$$\lambda_{1,2} = u \pm c \tag{3.7}$$

and right eigenvectors

$$r_1 = \begin{pmatrix} -\rho/c \\ 1 \end{pmatrix},\quad r_2 = \begin{pmatrix} \rho/c \\ 1 \end{pmatrix}. \tag{3.8}$$

Along the rarefaction curves defined by

$$(3.9)\quad \begin{aligned} \Gamma_1 &: \frac{d}{dt}\begin{pmatrix}\rho\\ u\end{pmatrix} = \begin{pmatrix}-\rho/c\\ 1\end{pmatrix} = r_1 \\ \Gamma_2 &: \frac{d}{dt}\begin{pmatrix}\rho\\ u\end{pmatrix} = \begin{pmatrix}\rho/c\\ 1\end{pmatrix} = r_2 \end{aligned}$$

the corresponding Riemann invariants are constant:

$$(3.10)\quad \begin{aligned} R_1 &= u + \frac{2c}{\gamma-1} = \text{constant on } \Gamma_1 \\ R_2 &= u - \frac{2c}{\gamma-1} = \text{constant on } \Gamma_2 \end{aligned}$$

To find the path of integration Γ^j , we first find the point of intersection in the c,u plane of the lines

$$(3.11)\quad \begin{aligned} L_1 &: u + \frac{2c}{\gamma-1} = u_j + \frac{2c_j}{\gamma-1} \\ L_2 &: u - \frac{2c}{\gamma-1} = u_{j-1} - \frac{2c_j}{\Gamma-1} \end{aligned}$$

Calling this point $(c_{j-1/2}, u_{j-1/2})$, we find

$$(3.12)\quad \begin{aligned} c_{j-1/2} &= \frac{c_j + c_{j-1}}{2} + \frac{\gamma-1}{4}(u_j - u_{j-1}) \\ u_{j-1/2} &= \frac{1}{\gamma-1}(c_j - c_{j-1}) + \frac{u_j + u_{j-1}}{2} . \end{aligned}$$

We need $c_{j-1/2} > 0$ for the scheme to make sense. Thus our only requirement is $u_{j-1} - u_j < \frac{2}{\gamma-1}(c_j + c_{j-1})$ in order that our scheme exist.

In order to construct the scheme, we need to parametrize Γ_1^j and Γ_2^j . We shall always do this using c as the parameter.

Along Γ_2^j we have:

$$(3.13)\quad \frac{d}{dc}\begin{pmatrix}\rho\\ u\end{pmatrix} = \begin{pmatrix}\frac{d\rho}{dc}\\ \frac{2}{\gamma-1}\end{pmatrix} = \frac{2}{\gamma-1}\begin{pmatrix}\frac{\rho}{c}\\ 1\end{pmatrix} = \frac{2}{\gamma-1} r_2(w) .$$

Hence:

(3.14)
$$\int_{\Gamma_2^j} (\partial f(\rho(c),u(c)))^+ dc = \frac{2}{\gamma-1}\int_{c_{j-1}}^{c_{j-\frac12}} \max(u(c)+c,0)\begin{pmatrix}\frac{\rho(c)}{c}\\ 1\end{pmatrix}dc$$

$$= \frac{2}{\gamma-1}\int_{c_{j-1}}^{c_{j-\frac12}} \max(u_{j-1} - \frac{2}{\gamma-1}c_{j-1} + \frac{\gamma+1}{\gamma-1}c,0)\begin{pmatrix}\frac{\rho(c)}{c}\\ 1\end{pmatrix}dc .$$

Similarly, along Γ_1^j, we have:

(3.15)
$$\frac{d}{dc}\begin{pmatrix}\rho\\ u\end{pmatrix} = \begin{pmatrix}\frac{d\rho}{dc}\\ \frac{-2}{\gamma-1}\end{pmatrix} = \frac{-2}{\gamma-1}\begin{pmatrix}-\frac{\rho}{c}\\ 1\end{pmatrix} = \frac{-2}{\gamma-1} r_1(w) ,$$

hence

(3.16)
$$\int_{\Gamma_1^j} (\partial f(\rho(c),u(c)))^+ dc = -\frac{2}{\gamma-1}\int_{c_{j-\frac12}}^{c_j} \max(u(c)-c,0)\begin{pmatrix}\frac{-\rho(c)}{c}\\ 1\end{pmatrix}dc$$

$$= -\frac{2}{\gamma-1}\int_{c_{j-\frac12}}^{c_j} \max((u_j + \frac{2}{\gamma-1}c_j - \frac{\gamma+1}{\gamma-1}c),0)\begin{pmatrix}\frac{-\rho(c)}{c}\\ 1\end{pmatrix}dc$$

In order to carry out the integration in (3.14) and (3.16), we first use the indefinite integral results:

(3.17)
$$\frac{2}{\gamma-1}\int (u_{j-1} - \frac{2}{\gamma-1}c_{j-1} + \frac{\gamma+1}{\gamma-1}c)\begin{pmatrix}\frac{\rho(c)}{c}\\ 1\end{pmatrix}dc$$

$$= (u_{j-1} - \frac{2}{\gamma-1}c_{j-1})\begin{pmatrix}\rho(c)\\ \frac{2}{\gamma-1}c\end{pmatrix} + \begin{pmatrix}\frac{2\rho(c)}{\gamma-1}c\\ \frac{\gamma+1}{(\gamma-1)^2}c^2\end{pmatrix}$$

and

(3.18)
$$\frac{-2}{\gamma-1}\int (u_j + \frac{2}{\gamma-1}c_j - \frac{\gamma+1}{\gamma-1}c)\begin{pmatrix}\frac{-\rho(c)}{c}\\ 1\end{pmatrix}dc$$

$$= (u_j + \frac{2}{\gamma-1}c_j)\begin{pmatrix}\rho(c)\\ \frac{-2}{\gamma-1}c\end{pmatrix} + \begin{pmatrix}\frac{-2}{\gamma-1}\rho(c)\ c\\ \frac{\gamma+1}{(\gamma-1)^2}c^2\end{pmatrix}$$

We may use these expressions in order to evaluate the integrals in (3.14) and (3.16), arriving at:

$$\text{(3.19)}\qquad \int_{\Gamma_2^j} (\partial f(\rho(c),u(c)))^+ dc = (u_{j-1} - \frac{2}{\gamma-1} c_{j-1})\begin{pmatrix} \tilde{\rho}_{j-1/2} - \tilde{\rho}_{j-1} \\ \frac{2}{\gamma-1}(\tilde{c}_{j-1/2} - \tilde{c}_{j-1}) \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{2}{\gamma-1} (\tilde{\rho}_{j-1/2}\tilde{c}_{j-1/2} - \tilde{\rho}_{j-1}\tilde{c}_{j-1}) \\ \frac{\gamma+1}{(\gamma-1)^2} (\tilde{c}^2_{j-1/2} - \tilde{c}^2_{j-1}) \end{pmatrix}$$

where $\tilde{\rho} = \rho(\tilde{c})$, and

(3.20) (a) if $\min(u_{j-1} + c_{j-1}, u_{j-1/2} + c_{j-1/2}) > 0$ then $\tilde{c}_{j-1/2} = c_{j-1/2}$, $\tilde{c}_{j-1} = c_{j-1}$

(b) if $u_{j-1} + c_{j-1} > 0 > u_{j-1/2} + c_{j-1/2}$

then $\tilde{c}_{j-1/2} = \bar{c}_{j-1/2} = -\frac{\gamma-1}{\gamma+1} (u_{j-1} - \frac{2}{\gamma+1} c_{j-1})$ and $\tilde{c}_{j-1} = c_{j-1}$.

(c) if $u_{j-1} + c_{j-1} < 0 < u_{j-1/2} + c_{j-1/2}$ then

$\tilde{c}_{j-1/2} = c_{j-1/2}$ and $\tilde{c}_{j-1} = \bar{c}_{j-1/2} = -(\frac{\gamma-1}{\gamma+1})(u_{j-1} - \frac{2}{\gamma-1} c_{j-1})$

(d) if $\max(u_{j-1} + c_{j-1}, u_{j-1/2} + c_{j-1/2}) < 0$ then $\tilde{c}_{j-1/2} = \tilde{c}_{j-1} = \bar{c}_{j-1/2}$,

(and in this case the quantity in (3.19) vanishes).

We have defined $\bar{c}_{j-1/2}$ so that on Γ_2^j it is true that $\bar{u}_{j-1/2} = -\bar{c}_{j-1/2}$, i.e. it is a sonic point.

We also have:

$$(3.21)\qquad \int_{\Gamma_1^j} (\partial f(\rho(c),u(c)))^+ dc = -\left(u_j + \frac{2}{\gamma-1} c_j\right)\begin{pmatrix} \tilde{\rho}_j - \tilde{\rho}_{j-\frac12} \\ -\frac{2}{\gamma-1}(\bar{c}_j - \bar{c}_{j-\frac12}) \end{pmatrix}$$

$$+ \begin{pmatrix} -\frac{2}{\gamma-1}\left(\bar{\rho}_j\bar{c}_j - \bar{\rho}_{j-\frac12}\bar{c}_{j-\frac12}\right) \\ \frac{\gamma+1}{(\lambda-1)^2}\left(\bar{c}_j^2 - \bar{c}_{j-\frac12}^2\right) \end{pmatrix}$$

where $\tilde{\rho} = \rho(\bar{c})$ and

(3.22) (a) if $\min(u_j - c_j,\ u_{j-\frac12} - c_{j-\frac12}) > 0$ then $\tilde{c}_{j-\frac12} = c_{j-\frac12},\ \bar{c}_j = c_j$

(b) if $u_j - c_j > 0 > u_{j-\frac12} - c_{j-\frac12}$

then $\tilde{c}_{j-\frac12} = \bar{c}_{j-\frac12} = \frac{\gamma-1}{\gamma+1}\left(u_j + \frac{2}{\gamma-1}c_j\right),\quad \tilde{c}_j = c_j$

(c) if $u_j - c_j < 0 < u_{j-\frac12} - c_{j-\frac12}$ then $\tilde{c}_j = \bar{c}_{j-\frac12},\ \bar{c}_{j-\frac12} = c_{j-\frac12}$

(d) if $\max(u_j - c_j,\ u_{j-\frac12} - c_{j-\frac12}) < 0$ then $\tilde{c}_j = \tilde{c}_{j-\frac12} = \bar{c}_{j-\frac12}$

(and in this case the quantity in (3.21) vanishes).

We have defined $\bar{c}_{j-\frac12}$ so that on Γ_1^j it is true that $\bar{u}_{j-\frac12} = \bar{c}_{j-\frac12}$, i.e. it is a sonic point.

Using divided difference operators we may now write out the explicit one step upwind difference approximation to (3.8):

$$(3.23)\qquad D_+^t \begin{pmatrix} \rho_j^n \\ u_j^n \end{pmatrix} = -D_+^x \begin{pmatrix} \rho_j^n\, u_j^n \\ \frac{(u_j^n)^2}{2} + \frac{(c_j^n)^2}{\gamma-1} \end{pmatrix} + D_+^x \lambda_+(\rho_j^n, u_j^n, \rho_{j-1}^n, u_{j-1}^n) =$$

$$= -\frac{1}{\Delta x}\int_{\Gamma^j} (\partial f(w^n))^+ dw - \frac{1}{\Delta x}\int_{\Gamma^{j+1}} (\partial f(w^n))^- dw$$

where

$$(3.24) \qquad A_+(\rho_j^n, u_j^n, \rho_{j-1}^n, u_{j-1}^n) = \left(u_{j-1}^n - \frac{2}{\gamma-1} c_{j-1}^n\right) \begin{pmatrix} \tilde{\rho}^n_{j-1/2} - \tilde{\rho}^n_{j-1} \\ \frac{2}{\gamma-1}(\tilde{c}^n_{j-1/2} - \tilde{c}^n_{j-1}) \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{2}{\gamma-1}\left(\tilde{\rho}^n_{j-1/2}\tilde{c}^n_{j-1/2} - \tilde{\rho}^n_{j-1}\tilde{c}^n_{j-1}\right) \\ \frac{\gamma+1}{(\gamma-1)^2}\left((\tilde{c}^n_{j-1/2})^2 - (\tilde{c}^n_{j-1})^2\right) \end{pmatrix}$$

$$+ \left(u_j^n + \frac{2}{\gamma-1} c_j^n\right) \begin{pmatrix} \tilde{\tilde{\rho}}^n_j - \tilde{\tilde{\rho}}^n_{j-1/2} \\ -\frac{2}{\gamma-1}(\tilde{\tilde{c}}^n_j - \tilde{\tilde{c}}^n_{j-1/2}) \end{pmatrix} + \begin{pmatrix} -\frac{2}{\gamma-1}\left(\tilde{\tilde{\rho}}^n_j \tilde{\tilde{c}}^n_j - \tilde{\tilde{\rho}}^n_{j-1/2}\tilde{\tilde{c}}^n_{j-1/2}\right) \\ \frac{\gamma+1}{(\gamma-1)^2}\left((\tilde{\tilde{c}}^n_j)^2 - (\tilde{\tilde{c}}^n_{j-1/2})^2\right) \end{pmatrix}$$

We may wish to use an implicit method based on the space differencing used in (3.23). Various possibilities exist. The most natural are

(3.25) (Fully implicit)

$$D_+^t \begin{pmatrix} \rho_j^n \\ u_j^n \end{pmatrix} = -\frac{1}{\Delta x} \int_{\Gamma^j} (\partial f(w^{n+1}))^+ dw - \frac{1}{\Delta x} \int_{\Gamma^{j+1}} (\partial f(w^{n+1}))^- dw$$

(3.26) (Crank-Nicolson)

$$2D_+^t \begin{pmatrix} \rho_j^n \\ u_j^n \end{pmatrix} = -\frac{1}{\Delta x} \int_{\Gamma^j} \left[(\partial f(w^n))^+ + (\partial f(w^{n+1}))^+\right] dw - \frac{1}{\Delta x} \int_{\Gamma^{j+1}} \left[\partial f(w^n))^- + (\partial f(w^{n+1}))^-\right] dw$$

We may rewrite (3.26) after multiplying the equation by Δt, as

$$(3.27) \qquad \begin{pmatrix} \rho_j^{n+1} \\ u_j^{n+1} \end{pmatrix} - \frac{\Delta t}{2\Delta x} T(\rho_j^{n+1}, u_j^{n+1}) = \begin{pmatrix} \rho_j^n \\ u_j^n \end{pmatrix} + \frac{\Delta t}{2\Delta x} T(\rho_j^n, u_j^n)$$

In order to invert the pair of non-linear equations for $\begin{pmatrix} \rho_j^{n+1} \\ u_j^{n+1} \end{pmatrix}$, we suggest using Newton's method with the initial guess $\begin{pmatrix} \rho_j^{n} \\ u_j^{n} \end{pmatrix}$, the solution at the previous time step. It is suggested by several authors, e.g. [1], [20], that only one iteration suffices in cases such as this.

Thus the full expression is:

$$(3.28)\qquad \begin{pmatrix} \rho_j^{n+1} \\ u_j^{n+1} \end{pmatrix} = \left(I - \frac{\Delta t}{2\Delta x}\, dT_{\rho^n,u^n}\right)^{-1}\left(\begin{pmatrix} \rho_j^{n} \\ u_j^{n} \end{pmatrix} + \frac{\Delta t}{2\Delta x}\, T(\rho_j^n, u_j^n)\right)$$

Of course in the regions where the Jacobian matrix has eigenvalues which are both positive or both negative the matrix $dT_{\underline{\rho},\underline{u}}$ is either upper or lower triangular. In particular, if

$$\underline{u}_{j+r} > \underline{c}_{j+r}\,,\quad r = 0, \pm 1$$

then

$$(3.29)\qquad dT_{\underline{\rho},\underline{u}} \begin{pmatrix} \rho_j \\ u_j \end{pmatrix} = -\Delta_- \begin{pmatrix} \underline{\rho}_j u_j + \underline{u}_j \rho_j \\ \underline{u}_j u_j + \dfrac{\underline{c}_j^2}{\underline{\rho}_j}\rho_j \end{pmatrix}$$

and if

$$\underline{u}_{j+r} < \underline{c}_{j+r}\,,\quad r = 0, \pm 1,$$

then:

$$(3.30)\qquad dT_{\underline{\rho},\underline{u}} \begin{pmatrix} \rho_j \\ u_j \end{pmatrix} = -\Delta_+ \begin{pmatrix} \underline{\rho}_j u_j + \underline{u}_j \rho_j \\ \underline{u}_j u_j + \dfrac{\underline{c}_j^2}{\underline{\rho}_j}\rho_j \end{pmatrix}$$

In general, a complicated, but fairly straightforward calculation gives us the linearized operator at a state $\underline{\rho}_j, \underline{u}_j$

$$
(3.31)\qquad dT_{\underline{\rho},\underline{u}}\begin{pmatrix}\rho_j\\ u_j\end{pmatrix} = -D_+^x\begin{pmatrix}\underline{\rho}_j u_j + \underline{u}_j\rho_j\\ \underline{u}_j u_j + \dfrac{\underline{c}_j^2}{\underline{\rho}_j}\rho_j\end{pmatrix} + D_+^x\Big[\max(\underline{u}_j - \underline{c}_j, 0)\begin{pmatrix}\rho_j\\ -\dfrac{\underline{c}_j}{\underline{\rho}_j}\rho_j\end{pmatrix}
$$

$$
-\max(\underline{u}_{j-1/2} - \underline{c}_{j-1/2}, 0)\begin{pmatrix}\dfrac{\underline{\rho}_{j-1/2}}{\underline{c}_{j-1/2}}\\ -1\end{pmatrix}\tfrac{1}{2}\left(\frac{\underline{c}_j}{\underline{\rho}_j}\rho_j + \frac{\underline{c}_{j-1}}{\underline{\rho}_{j-1}}\rho_{j-1} + u_j - u_{j-1}\right)
$$

$$
+\begin{pmatrix}\bar{\underline{\rho}}_j - \bar{\underline{\rho}}_{j-1/2}\\ -\dfrac{2}{\gamma-1}(\bar{\underline{c}}_j - \bar{\underline{c}}_{j-1/2})\end{pmatrix}\left(u_j + \frac{\underline{c}_j}{\underline{\rho}_j}\rho_j\right) + \max(\underline{u}_{j-1/2} + \underline{c}_{j-1/2})\begin{pmatrix}\dfrac{\underline{\rho}_{j-1/2}}{\underline{c}_{j-1/2}}\\ 1\end{pmatrix}\tfrac{1}{2}\left(\frac{\underline{c}_j}{\underline{\rho}_j}\rho_j + \frac{\underline{c}_{j-1}}{\underline{\rho}_{j-1}}\rho_{j-1} + u_j - u_{j-1}\right)
$$

$$
-\max(\underline{u}_{j-1} + \underline{c}_{j-1}, 0)\begin{pmatrix}1\\ \dfrac{\underline{c}_{j-1}}{\underline{\rho}_{j-1}}\end{pmatrix}\rho_{j-1} + \begin{pmatrix}\tilde{\underline{\rho}}_{j-1/2} - \tilde{\underline{\rho}}_{j-1}\\ \dfrac{2}{\gamma-1}(\tilde{\underline{c}}_{j-1/2} - \tilde{\underline{c}}_{j-1})\end{pmatrix}\left(u_{j-1} - \frac{\underline{c}_{j-1}}{\underline{\rho}_{j-1}}\rho_{j-1}\right)\Big]
$$

In the subsonic case when the eigenvalues of the Jacobian matrix ∂f are of both signs, the operator $dT_{\underline{\rho},\underline{u}}$ is a perturbation of a simple form. In particular if $-\underline{c}_{j+r} < \underline{u}_{j+r} < \underline{c}_{j+r}$, $r = 0, \pm 1$, then we define

$$
(3.32)\qquad dT_{\underline{\rho},\underline{u}}\begin{pmatrix}\rho_j\\ u_j\end{pmatrix} = -\sum_{r=-1}^{1} B_r^{(j)}(\underline{\rho},\underline{u})\begin{pmatrix}\rho_{j+r}\\ u_{j+r}\end{pmatrix}
$$

and we prove the following:

<u>Lemma (3.1)</u>

$$
B_{-1}^{(j)}(\underline{\rho},\underline{u}) = \bar{B}_{-1}(\underline{\rho}_{j-1},\underline{u}_{j-1}) + O(|\rho_j - \rho_{j-1}| + |u_j - u_{j-1}|)
$$

$$
B_1^{(j)}(\underline{\rho},\underline{u}) = \bar{B}_1(\underline{\rho}_{j+1},\underline{u}_{j+1}) + O(|\rho_{j+1} - \rho_j| + |u_{j+1} - u_j|)
$$

where

(a) $\bar{B}_{-1}$ and $\bar{B}_1$ are both of rank one.

(b) The eigenvalues of $\bar{B}_{-1}$ are 0 and $\underline{u}_{j-1} + c_{j-1} > 0$, with corresponding eigenvectors $\begin{pmatrix} -\frac{\underline{\rho}_{j-1}}{\underline{c}_{j+1}} \\ 1 \end{pmatrix}$, $\begin{pmatrix} \frac{\underline{\rho}_{j-1}}{\underline{c}_{j-1}} \\ 1 \end{pmatrix}$.

(c) The eigenvalues of $\bar{B}_1$ are $\underline{u}_{j+1} - \underline{c}_{j+1} < 0$, and 0 and the corresponding eigenvectors are $\begin{pmatrix} -\frac{\underline{\rho}_{j+1}}{\underline{c}_{j+1}} \\ 1 \end{pmatrix}$, $\begin{pmatrix} \frac{\underline{\rho}_{j+1}}{\underline{c}_{j+1}} \\ 1 \end{pmatrix}$.

As a consequence of this Lemma, we have the following:

<u>Remark (3.1)</u>

At a boundary point near which the flow is smooth, the effect of nonphysical and purely numerical boundary conditions is small (in fact zero in the supersonic case). This is true because of the previous Lemma since

$$\chi(\underline{\rho},\underline{u})\partial f(\underline{\rho},\underline{u}) = \bar{B}_{-1}(\underline{\rho},\underline{u})$$

$$(1 - \chi(\underline{\rho},\underline{u}))\partial f(\underline{\rho},\underline{u}) = \bar{B}_1(\underline{\rho},\underline{u})$$

(See (2.8) for the definition of $\chi(\underline{\rho},\underline{u})$).

<u>Proof of Lemma (3.1)</u>

We use (3.31) in this region to write

$$-B_{-1}^{(j)}(\underline{\rho},\underline{u})\begin{pmatrix} \rho_{j-1} \\ u_{j-1} \end{pmatrix} = \rho_{j-1}\Big[\tfrac{1}{2}\begin{pmatrix} \frac{\underline{\rho}_{j-1/2}}{\underline{c}_{j-1/2}} \\ 1 \end{pmatrix} \frac{\underline{c}_{j-1}}{\underline{\rho}_{j-1}}(\underline{u}_{j-1/2} + \underline{c}_{j-1/2}) \tag{3.33}$$

$$- (\underline{u}_{j-1} + \underline{c}_{j-1})\begin{pmatrix} 1 \\ \frac{\underline{c}_{j-1}}{\underline{\rho}_{j-1}} \end{pmatrix} - \frac{\underline{c}_{j-1}}{\underline{\rho}_{j-1}}\begin{pmatrix} \underline{\rho}_{j-1/2} - \underline{\rho}_{j-1} \\ \frac{2}{\gamma-1}(\underline{c}_{j-1/2} - \underline{c}_{j-1}) \end{pmatrix}\Big]$$

$$+ u_{j-1}\Big[-\tfrac{1}{2}\begin{pmatrix} \frac{\underline{\rho}_{j-1/2}}{\underline{c}_{j-1/2}} \\ 1 \end{pmatrix}(\underline{u}_{j-1/2} + \underline{c}_{j-1/2}) + \begin{pmatrix} \underline{\rho}_{j-1/2} - \underline{\rho}_{j-1} \\ \frac{2}{\gamma-1}(\underline{c}_{j-1/2} - \underline{c}_{j-1}) \end{pmatrix}\Big].$$

So

$$(3.34) \qquad \bar{B}_{-1}(\underline{\rho}_{j-1},\underline{u}_{j-1})\begin{pmatrix}\rho_{j-1}\\ u_{j-1}\end{pmatrix} = \rho_{j-1}\left[\begin{pmatrix}\frac{\underline{\rho}_{j-1}}{\underline{c}_{j-1}}\\ 1\end{pmatrix}\left(\tfrac{1}{2}\frac{\underline{c}_{j-1}}{\underline{\rho}_{j-1}}(\underline{u}_{j-1}+\underline{c}_{j-1})\right)\right]$$

$$+ u_{j-1}\left[\begin{pmatrix}\frac{\underline{\rho}_{j-1}}{\underline{c}_{j-1}}\\ 1\end{pmatrix}\left(\tfrac{1}{2}(\underline{u}_{j-1}+\underline{c}_{j-1})\right)\right]$$

and hence, finally:

$$(3.35) \qquad \bar{B}_{-1}(\underline{\rho}_{j-1},\underline{u}_{j-1}) = \tfrac{1}{2}(\underline{u}_{j-1}+\underline{c}_{j-1})\begin{pmatrix}1 & \frac{\underline{\rho}_{j-1}}{\underline{c}_{j-1}}\\ \frac{\underline{c}_{j-1}}{\underline{\rho}_{j-1}} & 1\end{pmatrix}$$

A simple calculation shows us that $\bar{B}_{-1}(\underline{\rho}_{j-1}, \underline{u}_{j-1})$ has one nontrivial eigenvalue equal to $\underline{u}_{j-1}+\underline{c}_{j-1}$, and corresponding eigenvectors equal to $\begin{pmatrix}\underline{\rho}_{j-1}/\underline{c}_{j-1}\\ 1\end{pmatrix}$. The zero eigenvalue has corresponding eigenvector $\begin{pmatrix}-\underline{\rho}_{j-1}/\underline{c}_{j-1}\\ 1\end{pmatrix}$.

An analogous proof works for $\bar{B}_1$.

4. THE ENTROPY CONDITION AND STEADY DISCRETE SHOCKS

In this section we shall prove results concerning entropy production and sharp, steady discrete shocks. These results are very much in the spirit of our earlier work [5], [7], with one major difference. These new results apply to a hyperbolic system, rather than a single conservation law. (See also Osher and Solomon [17]).

The entropy function for the hyperbolic system of conservation laws is:

$$V(\rho,u) = \frac{A\gamma\,\rho^{\gamma-1}}{(\gamma-2)(\gamma-1)} + \frac{u^2}{2} = \frac{c^2}{(\gamma-2)(\gamma-1)} + \frac{u^2}{2} . \tag{4.1}$$

In the Hessian matrix H is

$$H = \begin{pmatrix} A\gamma\,\rho^{\gamma-3} & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.2}$$

The additional conservation law is thus:

$$\frac{\partial V}{\partial t} + \frac{\partial}{\partial x} G = 0 , \tag{4.3}$$

with entropy flux function:

$$G = \frac{u^3}{3} + \frac{A\gamma\,\rho^{\gamma-1} u}{\gamma-2} . \tag{4.4}$$

In the special case of $\gamma = 2$, we replace this by $V = q(\rho) + \frac{u^2}{2}$, $G = u\rho q'(\rho) + \frac{u^3}{3}$, where q is any function satisfying $q'' = \frac{c^2}{\rho} = 2A$.

Equation (4.3) is valid only for continuous solutions of (3.5). Across discontinuities, and for general weak solutions, we impose the entropy inequality:

$$\frac{\partial}{\partial t} V + \frac{\partial}{\partial x} G \leq 0 . \tag{4.5}$$

It was shown by Lax [13], in a general setting, that for piecewise continuous solutions of (3.5), inequality (4.5) is equivalent to the geometric k shock condition for weak shocks. This was shown to be true also for strong shocks for a class of systems which includes (3.5). See Mock [15].

Approximations (3.23), (3.25), and (3.26) involve the same space discretization of (3.5). We may thus consider a semidiscrete approximation to (3.5) of the form:

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho_j \\ u_j \end{pmatrix} = -D_+^x \begin{pmatrix} \rho_j u_j \\ \frac{u_j^2}{2} + \frac{c_j^2}{\gamma-1} \end{pmatrix} + D_+^x A_+(\rho_j,u_j,\rho_{j-1},u_{j-1}) . \tag{4.6}$$

Steady solutions of (3.23), (3.25), (3.26), and (4.6) are all the same.

Our first theorem of this section is thus:

Theorem (4.1)

Suppose $w_j(t) = \begin{pmatrix} \rho_j(t) \\ u_j(t) \end{pmatrix}$ solves (4.6) and converges boundedly a.e. as $\Delta x \to 0$ to $w(x,t)$ with $\inf \rho_j(x,t) \geq \delta_1 > 0$. In addition, suppose the quantity $\lim\limits_{\Delta x \to 0} \sup\limits_{j,t} (|\Delta_+ \rho_j(t)| + |\Delta_+ u_j(t)|)$ is sufficiently small (but positive). Then w satisfies the entropy inequality (4.5).

Next, we present analytic evidence that our scheme gives excellent shock resolution in the steady case. (Numerical computations are presented in the next section both for the steady and unsteady case.).

Let $\begin{pmatrix} \rho^L \\ u^L \end{pmatrix}$, $\begin{pmatrix} \rho^R \\ u^R \end{pmatrix}$ be the states on the left and right (of $x = 0$ say) for a steady shock solution of (3.5). This means that both the jump conditions

$$\rho^L u^L = \rho^R u^R \tag{4.7}$$
$$\frac{(u^L)^2}{2} + \frac{(c^L)^2}{\gamma-1} = \frac{(u^R)^2}{2} + \frac{(c^R)^2}{\gamma-1}$$

and either the conditions for a one shock:

$$u^L > c^L \quad \text{and} \quad -c^R < u^R < c^R \tag{4.8}$$

or two shock:

$$-c^L < u^L < c^L \quad \text{and} \quad u^R < -c^R \tag{4.9}$$

are valid.

Then we seek solutions of (3.23), (3.25) or (3.26) which are time (or n) independent and approach $\begin{pmatrix} \rho^L \\ u^L \end{pmatrix}$ at $-\infty$ and $\begin{pmatrix} \rho^R \\ u^R \end{pmatrix}$ at $+\infty$. These are called steady discrete shocks.

We have

Theorem (4.2)

(i) Existence: A class of steady discrete shocks exists globally and each member is of

the following form:

$$\begin{pmatrix} \rho_j \\ u_j \end{pmatrix} \equiv \begin{pmatrix} \rho^L \\ u^L \end{pmatrix} \quad \text{for} \quad j < j_0 = 1 \,.$$

(4.10)

$$\begin{pmatrix} \rho_j \\ u_j \end{pmatrix} \equiv \begin{pmatrix} \rho^R \\ u^R \end{pmatrix} \quad \text{for} \quad j > j_0 + 1$$

and

$$\begin{pmatrix} \rho_{j_0}(\alpha) \\ u_{j_0}(\alpha) \end{pmatrix}, \quad \begin{pmatrix} \rho_{j_0+1}(\alpha) \\ u_{j_0+1}(\alpha) \end{pmatrix}$$

are a smooth one parameter family of states for $0 < \alpha < \alpha_r$, $r = 1,2$ with

(a) one shock:

$$u_{j_0}(\alpha) = \frac{-\rho_{j_0+1}(\alpha)u_{j_0+1}(\alpha) + \rho^R u^R + \bar{\bar{\rho}}\,\bar{\bar{c}}}{\rho_{j_0}(\alpha)}$$

$$u_{j_0+1}(\alpha) = u^R + \frac{2}{\gamma-1}\left(c^R - c_{j_0+1}(\alpha)\right)$$

$$\dot{c}_{j_0}(\alpha) = \frac{2}{\gamma-1}\left[u_{j_0+1}(\alpha) - c_{j_0+1}(\alpha)\right]\left[1 + \frac{u_{j_0}(\alpha)\,\rho_{j_0+1}(\alpha)}{\rho_{j_0}(\alpha)c_{j_0+1}(\alpha)}\right]$$

$$\dot{c}_{j_0+1}(\alpha) = \frac{2}{\gamma-1}\,\frac{c^2_{j_0}(\alpha) - u^2_{j_0}(\alpha)}{c_{j_0}(\alpha)}$$

($\cdot$ denotes $\frac{d}{d\alpha}$) with

$$\begin{pmatrix} c_{j_0}(0) \\ c_{j_0}(0) \end{pmatrix} = \begin{pmatrix} \bar{\bar{c}} \\ c^R \end{pmatrix}.$$

Here the solution of the system of differential equations is to be taken for $0 \leq \alpha \leq \alpha_1$, with

$$\begin{pmatrix} c_{j_0}(\alpha_1) \\ \bar{\bar{c}}_{j_0+1}(\alpha_1) \end{pmatrix} = \begin{pmatrix} c^L \\ \bar{\bar{c}} \end{pmatrix}$$

and $\bar{\bar{c}} = \frac{\gamma-1}{\gamma+1}(u^R + \frac{2}{\gamma-1} c^R)$.

(b) 2 shock:

$$u_{j_0} = u^L + \frac{2}{\gamma-1}(c_{j_0}(\alpha) - c^L)$$

$$u_{j_0+1}(\alpha) = \frac{-\rho_{j_0} u_{j_0} + \bar{\rho}\,\bar{c} + \rho^L u^L}{\rho_{j_0+1}(\alpha)}$$

$$\dot{c}_{j_0}(\alpha) = \frac{2}{\gamma-1} \frac{\left(c^2_{j_0+1}(\alpha) - u^2_{j_0+1}(\alpha)\right)}{c_{j_0+1}(\alpha)}$$

$$\dot{c}_{j_0+1}(\alpha) = \frac{2}{\gamma-1}(c_{j_0}(\alpha) + u_{j_0}(\alpha))\left(\frac{\rho_{j_0}(\alpha)u_{j_0+1}(\alpha)}{c_{j_0}(\alpha)\rho_{j_0+1}(\alpha)} - 1\right)$$

with

$$\begin{pmatrix} c_{j_0}(0) \\ c_{j_0+1}(0) \end{pmatrix} = \begin{pmatrix} c^L \\ \bar{c} \end{pmatrix} .$$

Here the solution of the differential equation is to be taken for $0 \leq \alpha \leq \alpha_2$ with

$$\begin{pmatrix} c_{j_0}(\alpha_2) \\ c_{j_0+1}(\alpha_2) \end{pmatrix} = \begin{pmatrix} \bar{c} \\ c^R \end{pmatrix} ,$$

and $\bar{c} = -\left(\frac{\gamma-1}{\gamma+1}\right)(u^L - \frac{2}{\gamma-1} c^L)$

(ii) Uniqueness:

We consider only steady discrete shocks having a certain weak monotonicity property.

(a) Admissible discrete steady one shocks are such that (i) $u_{j/2} + c_{j/2} > 0$ for all integers j, and (ii) if $u_j < c_j$ then $u_{j+1} < c_{j+1}$.

(b) Admissibile discrete steady two shocks are such that (i) $u_{j/2} - c_{j/2} < 0$ for all intergers j, and (ii) if $u_j < -c_j$, then $u_{j+1} < -c_{j+1}$.

Under these assumptions, all discrete steady shocks are of the form given in (i).

We note that without any of these restrictions, discrete steady shocks must be eventually constant. This means there exists integers $a < b$ such that $w_j \equiv w^L$ of $j < a$, and $w_j \equiv w^R$ of $j > b$, for some a,b. This is the content of Corollary (4.1) below.

We also believe that our uniqueness result is valid under weaker hypotheses - see the scalar result in [6], [7].

<u>Proof of Theorem (4.1)</u>

In its broad outline, this proof will follow the entropy inequality proofs in [6], [7]. There are various differences due to the vector valued nature of this problem.

Let $w_j(t)$ satisfy (4.6) and let $\varphi(x,t) \geq 0$ be in $C_0^1(R_+,R)$. Multiply (4.6) by $\varphi(x_j,t)\ V_w^T(w,t)\Delta x$, sum and integrate, and add $\sum_j \int \varphi(x_j,t)D_+^x G(w_j(t))dt$ to both sides. We then have

$$-\int dt \sum_j \Delta x[\varphi_t(x_j,t)V(w_j(t)) + (D_-^{(x)}\varphi(x_j,t))G(w_j(t))] \tag{4.11}$$

$$= \int dt \sum_j \Delta x\varphi(x_j,t)[\int_{\Gamma_{j+1}} V_w^T(w)[(\partial f(w))^+ + (\partial f(w))^- dw$$

$$- \int_{\Gamma_{j+1}} V_w^T(w_j)((\partial f(w))^-)dw - \int_{\Gamma_j} V_w^T(w_j)((\partial f(w))^+)dw] .$$

As $\Delta x \to 0$ the left side converges to

$$- \iint(\varphi_t V(w) + \varphi_x G(w))dxdt , \tag{4.12}$$

by the Lebesgue dominated convergence theorem. We must merely prove that the right side of (4.11) is nonpositive as $\Delta x \to 0$.

We rearrange the terms above, drop the t dependence, and arrive at

$$\text{(4.13)} \qquad \text{Right side of (4.11)} = \int dt \sum_j \Delta x \varphi(x_j) \int_{\Gamma_{j+1}} [V_w^T(w) - V_w^T(w_j)](\partial f(w)^-)dw$$

$$+ \int dt \sum_j \Delta x \varphi(x_j) \int_{\Gamma_{j+1}} (V_w^T(w) - V_w^T(w_{j+1})](\partial f(w))^+ dw$$

$$+ \int dt \sum_j \Delta x (D_+^x \varphi(x_{j-1})) \int_{\Gamma_j} V_w^T(w_j)(\partial f(w))^+ dw = [I] + [II] + [III] .$$

It is clear that:

$$\text{(4.14)} \qquad |[III]| \le K \int dt \sum_j \Delta x \, |\Delta_+ w_j| \varphi_x$$

where K depends on φ. Hence $[III] \to 0$ by the Lebesgue dominated convergence theorem.

Next we consider the integral along Γ_{j+1} in [I]

$$\text{(4.15)} \qquad \int_{\Gamma^{j+1}} (V_w^T(w) - V_w^T(w_j))\partial f(w)^- dw$$

$$= \int_{\Gamma_2^{j+1}} (V_w^T(w) - V_w^T(w_j))(\partial f(w))^+ dw + \int_{\Gamma_1^{j+1}} (V_w^T(w) - V_w^T(w_j))(\partial f(w))^- dw .$$

Now

$$\text{(4.16)} \qquad \int_{\Gamma_2^{j+1}} (V_w^T(w) - V_w^T(w_j))(\partial f(w))^- dw =$$

$$= \frac{2}{\gamma-1} \int_{c_j}^{c_{j+1/2}} dc \, \min(u_j - \frac{2}{\gamma-1} c_j + \frac{\gamma+1}{\gamma-1} c, 0) [\frac{1}{\gamma-2} (\frac{c^2}{\rho(c)} - \frac{c_j^2}{\rho_j}) \frac{\rho(c)}{c} + \frac{2}{\gamma-1} (c - c_j)$$

$$= (\frac{2}{\gamma-1})^2 \int_{c_j}^{c_{j+1/2}} dc \, \min(u_j - \frac{2}{\gamma-1} c_j + \frac{\gamma+1}{\gamma-1} c, 0) \left[\int_{c_j}^{c} [\frac{v}{\rho(v)} \frac{\rho(c)}{c} + 1] dv\right] .$$

Next we have:

$$\text{(4.17)} \qquad \int_{\Gamma_1^{j+1}} (V_w^T(w) - V_w^T(w_j))(\partial f(w))^- dw$$

$$= \frac{2}{\gamma-1} \int_{c_{j+1/2}}^{c_{j+1}} \min(u_{j+1} + \frac{2}{\gamma-1} c_{j+1} - \frac{\gamma+1}{\gamma-1} c, 0)[\frac{1}{\gamma-2}(\frac{c^2}{\rho(c)} - \frac{c_j^2}{\rho_j})\frac{\rho(c)}{c}$$

$$- (u_{j+1} - u_j + \frac{2}{\gamma-1}(c_{j+1} - c))dc$$

$$= \frac{2}{\gamma-1} \int_{c_{j+1/2}}^{c_j} \min(u_{j+1} + \frac{2}{\gamma-1} c_{j+1} - \frac{\gamma+1}{\gamma-1} c, 0)[\frac{1}{\gamma-1}(\frac{c^2}{\rho(c)} - \frac{c_{j+1/2}^2}{\rho_{j+1/2}}) \frac{\rho(c)}{c}$$

$$+ \frac{2}{\gamma-1}(c - c_{j+1/2}) + \frac{1}{\gamma-2}\left(\frac{c_{j+1/2}^2}{\rho_{j+1/2}} \frac{-c_j^2}{\rho_j^2}\right) \frac{\rho(c)}{c} - \frac{2}{\gamma-1}(c_{j+1/2} - c_j)]dc$$

$$= \frac{2}{\gamma-1} \int_{c_{j+1/2}}^{c_{j+1}} dc \max(u_{j+1} \frac{2}{\gamma-1} c_{j+1} - \frac{\gamma+1}{\gamma-1} c, 0 \left[\int_{c_{j+1/2}}^{c} (\frac{v}{\rho(v)} \frac{\rho(c)}{c} + 1)dv\right.$$

$$\left. + \tilde{K}(c, c_j, c_{j+1/2})(c_{j+1/2} - c_j)^2\right]$$

where $\tilde{K}$ and $\tilde{\tilde{K}}$ below are uniformly continuous functions.

Similar analysis gives us

$$\text{(4.18)} \qquad \int_{\Gamma_2^{j+1}} (V_w^T(w) - V_w^T(w_{j+1}))(\partial f(w))^+ dw$$

$$= -(\frac{2}{\gamma-1})^2 \int_{c_j}^{c_{j+1/2}} dc \max(u_j - \frac{2}{\gamma-1} c_j + \frac{\gamma+1}{\gamma-1} c, 0)\left[\int_{c}^{c_{j+1/2}} (\frac{v}{\rho(v)} \frac{\rho(c)}{c} + 1)dv\right.$$

$$\left. + \tilde{\tilde{K}}(c, c_{j+1}, c_{j+1/2})(c_{j+1} - c_{j+1/2})^2\right]$$

and

(4.19) $$\int_{\Gamma_1^{j+1}} (V_w^T(w) - V_w^T(w_{j+1}))(\partial f(w))^+ dw$$

$$= -\left(\frac{2}{\gamma-1}\right)^2 \int_{c_{j+1/2}}^{c_{j+1}} dc \max\left(u_{j+1} + \frac{2}{\gamma-1} c_{j+1} - \frac{\gamma+1}{\gamma-1} c, 0\right) \int_c^{c_{j+1}} \left(\frac{v}{\rho(v)} \frac{\rho(c)}{c} + 1\right) dv .$$

We add the last four equations and arrive at:

(4.20) $$\int_{\Gamma^{j+1}} [V_w^T(w) - V_w^T(w_j)](\partial f(w))^- dw + \int_{\Gamma^{j+1}} [V_w^T(w) - V_w^T(w_{j+1})](\partial f(w))^+ dw <$$

$$< -\frac{2}{(\gamma-1)^2} [-\min(u_j^{(1)} + c_j^{(1)}, 0) + \max(u_j^{(2)} + c_j^{(2)}, 0)](c_{j+1/2} - c_j)^2$$

$$+ a \max(u_j^{(2)} + c_j^{(2)}, 0)|c_{j+1/2} - c_j||c_{j+1} - c_{j+1/2}|^2$$

$$- \frac{2}{(\gamma-1)^2} [-\min(u_j^{(3)} - c_j^{(3)}, 0) + \max(u_j^{(4)} - c_j^{(4)}, 0)](c_{j+1} - c_{j+1/2})^2$$

$$- a \min(u_j^{(4)} - c_j^{(4)}, 0)|c_{j+1/2} - c_j|^2|c_{j+1} - c_{j+1/2}| .$$

Here a is a universal positive constant and the $u_j^{(r)} < c_j^{(r)}$ are evaluated somewhere on the relevant above paths of integration.

We first note a simple fact:

(4.21)
$$-\min(u_j^{(1)} + c_j^{(1)}, 0) + \max(u_j^{(2)} + c_j^{(2)}, 0)$$
$$-\min(u_j^{(3)} - c_j^{(3)}, 0) + \max(u_j^{(4)} - c_j^{(4)}, 0) > \delta > 0$$

for δ a universal constant.

The right side of (4.20) is obviously nonpositive for $|\Delta_+ w_j| < \varepsilon$, except perhaps in the following cases: either

(a) $$\inf_{c \in \Gamma_2^{j+1}} \left|u_j - \frac{2}{\gamma-1} c_j + \frac{\gamma+1}{\gamma-1} c\right| < \sup_{c \in \Gamma_j^{j+1}} \left|u_{j+1} + \frac{2}{\gamma-1} c_{j+1} - \frac{\gamma+1}{\gamma-1} c\right| |c_{j+1} - c_{j+1/2}|$$

or

(b) $$\inf_{c \in \Gamma_1^{j+1}} |u_{j+1} + \frac{2}{\gamma-1} c_{j+1} - \frac{\gamma+1}{\gamma-1} c| < \sup_{c \in \Gamma_2^{j+1}} |u_j - \frac{2}{\gamma-1} c_j + \frac{\gamma+1}{\gamma-1} c||c_{j+1/2} - c_j| .$$

However when (a) happens, it is easy to dominate (4.20) for ε sufficiently small.

(4.22) $$\int_{\Gamma^{j+1}} |v_w^T(w) - v_w^T(w_j)|(\partial f(w))^- dw + \int_{\Gamma^{j+1}} |v_w^T(w) - v_w^T(w_{j+1})|(\partial f(w))^+ dw$$

$$< -b_1|c_{j+1/2} - c_j|^3 - b_2|c_{j+1} - c_{j+1/2}|^2 + b_3|c_{j+1/2} - c_j|^2|c_{j+1} - c_{j+1/2}|^2$$

$$+ b_4|c_{j+1/2} - c_j||c_{j+1} - c_{j+1/2}|$$

for the b_j universal positive constants.

It is easy to show that this expression is nonpositive for sufficiently small ε .

A similar argument follows when (b) occurs. Thus the expressions [I] + [II] is nonpositive and we are finished.

<u>Proof of Theorem (4.2)</u>

We begin by demonstrating that the functions in (4.10) are indeed steady discrete shock solutions of (3.23), (3.25), or (3.26). This means that the sequence $w_j = \begin{pmatrix} \rho_j \\ u_j \end{pmatrix}$ solves the following: for each j

(4.23) $$\int_{\Gamma^j} (\partial f(w))^+ dw + \int_{\Gamma^{j+1}} (\partial f(w))^- dw = 0 .$$

This is trivially valid for $j < j_0 - 2$, $j > j_0 + 3$.

We must merely verify that

(4.24) (a) $$\int_{\Gamma^{j_0}} (\partial f(w))^- dw = 0$$

(b) $$\int_{\Gamma^{j_0}} (\partial f(w))^+ dw + \int_{\Gamma^{j_0+1}} (\partial f(w))^- dw = 0$$

(c) $$\int_{\Gamma^{j_0+1}} (\partial f(w))^+ dw + \int_{\Gamma^{j_0+2}} (\partial f(w))^- dw = 0$$

(d) $$\int_{\Gamma^{j_0+2}} (\partial f(w))^+ dw = 0 \ .$$

We need only verify any three of these, the fourth will then be valid automatically. This follows by merely summing all four, and using the shock jump relation $f(u^L) = f(u^R)$.

We first consider the discrete one shock case.

In order that (4.24)(a) be satisfied, we need $u > -c$ on the line segment connecting (c^L, u^L) to $(c_{j_0} - \frac{1}{2}, u_{j_0} - \frac{1}{2})$. (We shall always require that $c_{j_0} + \frac{1}{2} > 0$ for $v = -1,0,1,2,3$ so that cavitation does not take place.) we also require that $u > c$ on the line segment connecting $(c_{j_0 - \frac{1}{2}}, u_{j_0 - \frac{1}{2}})$ to $c_{j_0}, u_{j_0})$. This makes the condition above redundant. Thus for (a) we need only $u_{j_0} > c_{j_0} > 0$, $u_{j_0 - \frac{1}{2}} > c_{j_0 - \frac{1}{2}} > 0$.

For (d) to be valid, we first require that $(c_{j_0+1}, u_{j_0+1}) = (c_{j_0+3/2}, u_{j_0+3/2})$.

This means that (c_{j_0+1}, u_{j_0+1}) is connected to (c^R, u^R) via a one wave - i.e.

$u_{j_0+1} + \frac{2}{\gamma-1} c_{j_0+1} = u^R + \frac{2}{\gamma-1} c^R$. Finally, for (d), we require $u < c_{j_0+1} > u_{j_0+1}$.

We now must verify equation (c). Since (c_{j_0+1}, u_{j_0+1}) is connected to (c^R, u^R) by a subsonic one wave, we have:

$$\int_{\Gamma^{j_0+1}} (\partial f(w))^{-} dw = f(w^R) - f(w_{j_0+1})$$

(4.25)

$$= \begin{pmatrix} \rho^R u^R - \rho_{j_0+1} u_{j_0+1} \\ \frac{(u^R)^2}{2} + \frac{(c^R)^2}{\gamma-1} - \frac{u^2_{j_0+1}}{2} - \frac{c^2_{j_0+1}}{\gamma-1} \end{pmatrix} .$$

Next we require that $u_{j_0+\frac{1}{2}} > c_{j_0+\frac{1}{2}} > 0$, which implies, among other things:

(4.26) $$\int_{\Gamma^{j_0+1}} (\partial f(w))^{+} dw = f(\bar{\bar{w}}_{j_0+\frac{1}{2}}) - f(w_{j_0})$$

where $$\bar{\bar{w}}_{j_0+\frac{1}{2}} = (\bar{\bar{c}}_{j_0+\frac{1}{2}}, \bar{\bar{c}}_{j_0+\frac{1}{2}})$$

$$= \left(\frac{\gamma-1}{\gamma+1} \right)\left(u_{j_0+1} + \frac{2}{\gamma-1} c_{j_0+1}\right)(1,1)$$

$$= \frac{\gamma-1}{\gamma+1} \left(u^R + \frac{2}{\gamma-1} c^R\right) (1,1)$$

(see equation (3.21)(b).).

Thus to verify that we indeed have a steady discrete shock, we need to show that

(4.27) $$f(w^R) + f(\bar{\bar{w}}_{j_0+\frac{1}{2}}) - f(w_{j_0}) - f(w_{j_0+1}) = 0 ,$$

or:

(4.28) $$\begin{pmatrix} \bar{\bar{\rho}}_{j_0+\frac{1}{2}} \bar{\bar{c}}_{j_0+\frac{1}{2}} - \rho_{j_0} u_{j_0} + \rho^R u^R - \rho_{j_0+1} u_{j_0+1} \\ \frac{\bar{\bar{c}}^2_{j_0+\frac{1}{2}}}{2} + \frac{\bar{\bar{c}}^2_{j_0+\frac{1}{2}}}{\gamma-1} - \frac{u^2_{j_0}}{2} - \frac{c^2_{j_0}}{\gamma-1} + \frac{(u^R)^2}{2} + \frac{(c^R)^2}{\gamma-1} - \frac{u^2_{j_0+1}}{2} - \frac{u^2_{j_0+1}}{\gamma-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

subject to the restrictions that

(4.29) (a) $u_{j_0-\frac{1}{2}} > c_{j_0-\frac{1}{2}} > 0$

(b) $u_{j_0} > c_{j_0} > 0$

(c) $u_{j_0+\frac{1}{2}} > c_{j_0+\frac{1}{2}} > 0$

(d) $-c_{j_0+1} < u_{j_0+1} < c_{j_0+1} > 0$

(e) $u_{j_0+1} + \frac{2}{\gamma-1} c_{j_0+1} = u^R + \frac{2}{\gamma-1} c^R$.

Next we notice that the jump conditions (4.7) are equivalent to:

(4.30) (a) $$\frac{(\rho^R u^R)^2}{2(\rho^L)^2} + \frac{(c^L)^2}{\gamma-1} = \frac{(u^R)^2}{2} + \frac{(c^R)^2}{\gamma-1},$$

with

(b) $$u^L = \frac{\rho^R u^R}{\rho^L} .$$

The derivative of the function on the left in (4.30)(a) with respect to c^L is $\frac{2}{\gamma-1} (\rho^L c^L)^{-2}((\rho^L c^L)^2 - (\rho^R u^R)^2)$. Thus, a single minimum occurs when $u^L = c^L = (u^R)^{\frac{\gamma-1}{\gamma+1}} (c^R)^{\frac{2}{\gamma+1}}$. For this value, the quantity on the left in (4.30) is

$$\frac{\gamma+1}{2(\gamma-1)} (c^R)^{\frac{4}{\gamma+1}} (u^R)^{2\left(\frac{\gamma-1}{\gamma+1}\right)} < \frac{1}{\gamma-1} (c^R)^2 + \tfrac{1}{2} (u^R)^2 ,$$

by Holder's inequality. Thus, the jump conditions have one solution (c^L, u^L) for each (c^R, u^R), and for $0 < u^R < c^R$, we have $u^L > c^L > 0$. This solution is characterized by the root of (4.30)(a) satisfying

$$0 < c^L < (c^R)^{\frac{2}{\gamma+1}} (u^R)^{\frac{\gamma-1}{\gamma+1}} .$$

This analysis is needed in what follows.

Solving the first equation in (4.28) for u_{j_0} gives

$$u_{j_0} = -\frac{\rho_{j_0+1}(u^R + \frac{2}{\gamma-1}(c^R - c_{j_0+1})) + \rho^R u^R + \bar{\bar{\rho}}\,\bar{\bar{c}}}{\rho_{j_0}}, \tag{4.31}$$

(dropping the $j_0 + \frac{1}{2}$ subscript, both above and in what follows).

The second equation in (4.31) then becomes

$$0 = -\frac{\left(\rho_{j_0+1}(u^R + \frac{2}{\gamma-1}(c^R - c_{j_0+1})) - \rho^R u^R - \bar{\bar{\rho}}\,\bar{\bar{c}}\right)^2}{2\rho_{j_0}^2}$$

$$-\frac{c_{j_0}^2}{\gamma-1} + \frac{\gamma+1}{2(\gamma-1)}(\bar{\bar{c}})^2 + \frac{(u^R)^2}{2} + \frac{(c^R)^2}{\gamma-1} - \frac{c_{j_0+1}}{\gamma-1} - \frac{1}{2}(u^R + \frac{2}{\gamma-1}(c^R - c_{j_0+1}))^2$$
$$= g(c_{j_0}, c_{j_0+1}), \tag{4.32}$$

thus defining the function $g(c_{j_0}, c_{j_0+1})$.

One solution to this equation is $(c_{j_0}, c_{j_0+1}) = (\bar{\bar{c}}, c^R)$, another is $(c^L, \bar{\bar{c}})$. We shall show that there exists a one parameter family of solutions $(c_{j_0}(\alpha), c_{j_0+1}(\alpha))$, with both components increasing monotonically from $(\bar{\bar{c}}, c^R)$ to $(c^L, \bar{\bar{c}})$.

We shall then demonstrate that (4.29)(a) - (d) is valid for these solutions, ((e) is immediate from the construction).

A straightforward calculation, using (4.31) and (4.32), gives as:

(4.33) (a)
$$\frac{\partial}{\partial c_{j_0}} g = g_{c_{j_0}} = \frac{2}{(\gamma-1)c_{j_0}}(u_{j_0}^2 - c_{j_0}^2)$$

(b)
$$g_{c_{j_0+1}} = \frac{2}{\gamma-1}(u_{j_0+1} - c_{j_0+1})\left(1 + \frac{u_{j_0}\rho_{j_0+1}}{\rho_{j_0}c_{j_0+1}}\right)$$

We consider the system of ordinary differential equations

(4.34) (a) $$\frac{dc_{j_0}(\alpha)}{d\alpha} = g_{c_{j_0+1}}(c_{j_0}, c_{j_0+1})$$

$$\frac{dc_{j_0+1}(\alpha)}{d\alpha} = -g_{c_{j_0}}(c_{j_0}, c_{j_0+1})$$

with initial conditions

(b) $$\begin{pmatrix} c_{j_0}(0) \\ c_{j_0+1}(0) \end{pmatrix} = \begin{pmatrix} \bar{\bar{c}} \\ c^R \end{pmatrix}$$

to be solved for $\alpha > 0$.

Since the initial data solves (4.32), it is clear that the solution of (4.34) will also solve (4.32). We shall now show that $g_{c_{j_0}} > 0$, $g_{c_{j_0+1}} < 0$, for

$\bar{\bar{c}} < c_{j_0}(\alpha) < c^L$, $c^R < c_{j_0+1}(\alpha) < \bar{\bar{c}}$, with $g_{c_{j_0}} = 0$ only at the left endpoint

$c_{j_0} = \bar{\bar{c}}$, $g_{c_{j_0+1}} = 0$ only at the right endpoint $c_{j_0+1} = \bar{\bar{c}}$.

The statement concerning $g_{c_{j_0+1}}$ is simple to verify since by (4.32)(e),

$u_{j_0+1} = c_{j_0+1} \iff c_{j_0+1} = \bar{\bar{c}}$, and $g_{c_{j_0+1}} < 0$ at $\alpha = 0$.

In order to prove the statement concerning $g_{c_{j_0}}$, we shall use $\rho_{j_0} u_{j_0} > \rho_{j_0} c_{j_0}$

for all the relevant values of α except $\alpha = 0$.

This is equivalent to showing: for these values of α:

(4.35) $$\rho_{j_0}(u_{j_0} - c_{j_0}) = \rho^R u^R + \bar{\bar{\rho}}\,\bar{\bar{c}} - (\rho_{j_0} c_{j_0} + \rho_{j_0+1} u_{j_0+1}) > 0 .$$

Equality holds for $\alpha = 0$.

Using (4.31) - (4.34), we differentiate the left side of (4.35) and obtain:

$$\frac{d}{d\alpha}\rho_{j_0}(u_{j_0} - c_{j_0}) = -\left(\frac{\gamma+1}{\gamma-1}\right)\rho_{j_0}\dot{c}_{j_0} - \frac{2}{\gamma-1}\dot{c}_{j_0+1}\frac{\rho_{j_0+1}}{c_{j_0+1}}(u_{j_0+1} - c_{j_0+1})$$

$$= \frac{2}{\gamma-1}(c_{j_0+1} - u_{j_0+1})\left[\frac{2}{(\gamma-1)c_{j_0}}(c_{j_0}^2 - u_{j_0}^2)\frac{\rho_{j_0+1}}{c_{j_0+1}} + \frac{\gamma+1}{\gamma-1}\rho_{j_0}\left(1 + \frac{u_{j_0}}{\rho_{j_0}}\frac{\rho_{j_0+1}}{c_{j_0+1}}\right)\right] .$$

Thus we have the inequality:

(4.36) $$\frac{d}{d\alpha}\rho_{j_0}(u_{j_0} - c_{j_0}) + K_1(\alpha)\rho_{j_0}(u_{j_0} - c_{j_0}) < -K_2(\alpha)$$

with $K_1, K_2 > 0$ as long as $u_{j_0+1} < c_{j_0+1}$.

This means that $u_{j_0}(\alpha) > c_{j_0}(\alpha) > 0$ as long as $u_{j_0+1}(\alpha) < c_{j_0+1}(\alpha)$.

Thus the solution of (4.34) exists inside $c^L > c_{j_0} > \bar{\bar{c}}$, $\bar{\bar{c}} > c_{j_0+1} > c^R$. We wish to show it escapes at $(c^L, \bar{\bar{c}})$. When the solution passes through $(c_{j_0}(\alpha_1), \bar{\bar{c}})$, as it must for some α_1 , equations (4.31), (4.32) are valid with $u_{j_0} > c_{j_0}$, $u_{j_0+1} = c_{j_0+1} = \bar{\bar{c}}$.

These equations then become the jump conditions, (4.30), already analyzed, with (ρ_{j_0}, u_{j_0}) taking th place of (ρ^L, u^L) . Thus the unique solution must be $c_{j_0}(\alpha_1) = c^L$.

We must now merely verify inequalities (4.29)(a) and (c). ((b) and (d) are immediate from the construction 1.

First, we note from (3.12), that

(4.37) $$c_{j_0-\frac{1}{2}} = \frac{c^L + c_{j_0}}{2} + \frac{\gamma-1}{4}(u_{j_0} - u^L) .$$

Thus, using (4.31) - (4.34), we have

$$(4.38)\ \frac{d}{d\alpha} c_{j_0-\frac12} = \tfrac12 \frac{dc_{j_0}}{d\alpha} + \frac{\gamma-1}{4}\frac{du_{j_0}}{d\alpha} = \tfrac12 \frac{dc_{j_0}}{d\alpha}\left(1 - \frac{u_{j_0}}{c_{j_0}}\right) - \tfrac12 \frac{dc_{j_0+1}}{d\alpha}\frac{\rho_{j_0+1}}{\rho_{j_0}}\left(1 - \frac{u_{j_0+1}}{\rho_{j_0+1}}\right)$$

$$= \frac{1}{\gamma-1}(u_{j_0+1} - c_{j_0+1})\frac{(c_{j_0} - u_{j_0})}{c_{j_0}}\left(1 + \frac{u_{j_0}\rho_{j_0+1}}{\rho_{j_0}c_{j_0+1}} - \frac{\rho_{j_0+1}}{\rho_{j_0}c_{j_0+1}}(c_{j_0} + u_{j_0})\right)$$

$$= \frac{1}{\gamma-1}(u_{j_0+1} - c_{j_0+1}((\frac{c_{j_0} - u_{j_0}}{c_{j_0}})(1 - \frac{c_{j_0}}{\rho_{j_0}}\frac{\rho_{j_0+1}}{c_{j_0+1}}) .$$

This quantity is easily shown to be nonnegative if $1 < \gamma \leq 3$. Hence the minimum of $c_{j_0-\frac12}$ occurs at $u_{j_0} = c_{j_0} = \bar{\bar{c}}$.

We substitute that into (4.37) and we must show

$$(4.39)\qquad \frac{c^L + c^R}{2} + \frac{\gamma-1}{4}(u^R - u^L) > 0 .$$

This means we must verify:

$$(4.40)\qquad \frac{2c^L}{\gamma-1} + \frac{2c^R}{\gamma-1} + u^R > u^L = \sqrt{(u^R)^2 + \frac{2(c^R)^2}{\gamma-1} - \frac{2(c^L)^2}{\gamma-1}}$$

(using the second jump condition in (4.7)). This is trivially valid if $\frac{2}{\gamma-1} \geq 1$, or $\alpha \leq 3$.

Next we note, again using (3.12), that

$$(4.41)\qquad u_{j_0-\frac12} = \frac{1}{\gamma-1}(c_{j_0} - c^L) + \frac{u_{j_0}}{2} + u^L ,$$

and

$$(4.42)\qquad \frac{d}{d\alpha}u_{j_0-\frac12} = \frac{1}{\gamma-1}\frac{dc_{j_0}}{d\alpha} + \tfrac12\frac{du_{j_0}}{d\alpha} = \frac{2}{\gamma-1}\frac{dc_{j_0-\frac12}}{d\alpha} .$$

Thus:

$$(4.43) \qquad \frac{d}{d\alpha}(u_{j_0-1/2} - c_{j_0-1/2}) = \frac{3-\gamma}{(\gamma-1)^2}(u_{j_0+1} - c_{j_0+1})\left(\frac{(c_{j_0} - u_{j_0})}{c_{j_0}}\right)\left(1 - \frac{c_{j_0}\,\rho_{j_0+1}}{\rho_{j_0} c_{j_0+1}}\right).$$

This quantity is (compare with (4.38)) nonnegative for $1 < \gamma \leq 3$. Hence the minimum of $u_{j_0-1/2} - c_{j_0-1/2}$ occurs for $u_{j_0} = c_{j_0} = \bar{\bar{c}}$. We substitute this value into (4.37), and must show:

$$(4.44) \qquad (\gamma+1)(u^L - \frac{2}{\gamma-1}c^L) > (\gamma-3)(u^R + \frac{2}{\gamma-1}c^R) ,$$

but we have already shown:

$$(4.45) \qquad c^L < (c^R)^{\frac{2}{\gamma+1}}(u^R)^{\frac{\gamma-1}{\gamma+1}} \Longrightarrow u^L > (c^R)^{\frac{2}{\gamma+1}}(u^R)^{\frac{\gamma-1}{\gamma+1}}$$

and thus, by Hölder's inequality and the fact that $\gamma \leq 3$

$$(4.46) \qquad (\gamma+1)(u^L - \frac{2}{\gamma-1}c^L) > (\gamma+1)\left(\frac{\gamma-3}{\gamma-1}\right)(c_R)^{\frac{2}{\gamma+1}}(u^R)^{\frac{\gamma-1}{\gamma+1}} > (\gamma-3)(u^R + \frac{2}{\gamma-1}c^R) .$$

Next we note that:

$$(4.47) \qquad \begin{aligned} c_{j_0+1/2} &= \frac{c_{j_0+1} + c_{j_0}}{2} + \frac{\gamma-1}{4}(u_{j_0+1} - u_{j_0}) \\ &= \frac{c^R}{2} + \frac{\gamma-1}{4}u^R + \frac{c_{j_0}}{2} - \frac{\gamma-1}{4}u_{j_0} \end{aligned}$$

by (4.29)(c).

Thus, using (4.31) - (4.34), and (4.38), we have:

$$\frac{d}{d\alpha} c_{j_0+\frac{1}{2}} = \tfrac{1}{2}\frac{d}{d\alpha} c_{j_0} - \frac{\gamma-1}{4} \frac{d}{d\alpha} u_{j_0} \tag{4.48}$$

$$= \frac{dc_{j_0}}{d\alpha} - \frac{1}{\gamma-1}(u_{j_0+1} - c_{j_0+1}) \frac{(c_{j_0} - u_{j_0})}{c_{j_0}} \left(1 - \frac{c_{j_0}}{\rho_{j_0}} \frac{\rho_{j_0+1}}{c_{j_0+1}}\right)$$

$$= -\frac{1}{\gamma-1} (u_{j_0+1} - c_{j_0+1})\left(-2 - \frac{2u_{j_0}\rho_{j_0+1}}{\rho_{j_0}c_{j_0+1}} + 1 - \frac{u_{j_0}}{c_{j_0}} - \frac{c_{j_0}\rho_{j_0+1}}{\rho_{j_0}c_{j_0+1}} + \frac{u_{j_0}\rho_{j_0+1}}{\rho_{j_0}c_{j_0+1}}\right)$$

$$= \frac{1}{\gamma-1} (u_{j_0+1} - c_{j_0+1}) \frac{(u_{j_0} + c_{j_0})}{c_{j_0}} \left(1 + \frac{\rho_{j_0+1}c_{j_0}}{\rho_{j_0}c_{j_0+1}}\right) < 0 .$$

Thus, the minimum value of $c_{j_0+\frac{1}{2}}$ occurs at $(c_{j_0}, u_{j_0}) = (c^L, u^L)$, at which $c_{j_0+\frac{1}{2}} = \frac{c^R + c^L}{2} + \frac{\gamma-1}{4} (u^R u^L)$, which we showed was positive in (4.40).

Next we note that

$$u_{j_0+\frac{1}{2}} - c_{j_0+\frac{1}{2}} = \frac{1}{\gamma-1} (c_{j_0+1} - c_{j_0}) + \frac{u_{j_0+1} + u_{j_0}}{2} \tag{4.49}$$

$$- \frac{(c_{j_0+1} + c_{j_0})}{2} - \frac{\gamma-1}{4} (u_{j_0+1} - u_{j_0})$$

$$= \frac{3-\gamma}{4} \left(u^R + \frac{2}{\gamma-1} c^R\right) + \frac{\gamma+1}{4} \left(u_{j_0} - \frac{2}{\gamma-1} c_{j_0}\right) ,$$

by (4.29)(e). Hence by (4.48)

$$\frac{d}{d\alpha} (u_{j_0+\frac{1}{2}} - c_{j_0+\frac{1}{2}}) = - \frac{\gamma+1}{\gamma-1} \frac{d}{d\alpha} c_{j_0+\frac{1}{2}} > 0 . \tag{4.50}$$

Thus the minimum of $u_{j_0+\frac{1}{2}} - c_{j_0+\frac{1}{2}}$ occurs at $(c_{j_0}, u_{j_0}) = (\bar{\bar{c}}, \bar{\bar{c}})$, where

$$\min(u_{j_0+\frac{1}{2}} - c_{j_0+\frac{1}{2}}) = 0 . \tag{4.51}$$

We have thus verified that each member of the relevant family of sequences defined in the statement of Theorem (4.2) is indeed a discrete one shock.

An analogous verification works in the discrete two shock case.

We finally turn to the proof of uniqueness - that these are the only steady discrete shock solutions of (3.23), (3.25), or (3.26).

We begin with

<u>Lemma (4.1)</u>

Suppose

$$\int_{\Gamma^j} (\partial f(w))^+ dw + \int_{\Gamma^{j+1}} (\partial f(w))^- dw = 0$$

for $j = j_0 , j_0 + 1$. Moreover, suppose none of the eigenvalues of $\partial f(w_j)$ either vanish or change sign for $j = j_0-1, j_0 , j_0+1, j_0+2$. Then $w_{j_0} = w_{j_0+1}$.

<u>Proof</u>

Suppose first the eigenvalues of $\partial f(w)$ are both positive. Then $f(w_{j_0}) = f(w_{j_0-1})$ and by evaluating (4.52) for $j = j_0$, we have

$$f(w_{j_0}) - f(w_{j_0-1}) = \int_{\Gamma_2^{j_0}} \partial f(w)dw + \int_{\Gamma_2^{j_0}} \partial f(w)dw \tag{4.53}$$

$$= \underline{\gamma}_{j_0}(\underline{u}_{j_0} + \underline{c}_{j_0})r_2(w_{j_0}) + \underline{\underline{\gamma}}_{j_0}(\underline{\underline{u}}_{j_0} - \underline{\underline{c}}_{j_0})r_1(w_{j_0}) = 0 ,$$

for some intermediate values $\underline{w}_{j_0}$, $\underline{\underline{w}}_{j_0}$, and with $\underline{\gamma}_{j_0} = |w_{j_0-1/2} - w_{j_0-1}|\underline{c}_{j_0}$, $\underline{\underline{\gamma}}_{j_0} = |w_j - w_{j_0-1/2}|\underline{\underline{c}}_{j_0}$, with $\underline{c}_{j_0}$, $\underline{\underline{c}}_{j_0}$ bounded above and below by positive constants. This follows from the mean value theorem for integrals. The vectors $r_1(\underline{\underline{w}}_{j_0})$, $r_2(\underline{w}_{j_0})$ are of the form $\binom{-a}{1}$, $\binom{b}{1}$ for $a,b > 0$, hence they are linearly independent, and it follows that $w_{j_0-1} = w_{j_0-1/2} = w_{j_0}$. Thus if all the eigenvalues are positive, then $w_{j_0-1} = w_{j_0} = w_{j_0+1}$.

If all the eigenvalues are negative, then an analogous argument gives $w_{j_0} = w_{j_0+1} = w_{j_0+2}$.

Next suppose the eigenvalues are of different signs, i.e. $-c_j < u_j < c_j$, $j = j_0 + r-1$, $r = 0,1,2,3$. Then the mean value theorem for integrals allows us to rewrite (4.52) as

(4.54) $$\underline{\gamma}_j(\underline{u}_j + \underline{c}_j)r_2(\underline{w}_j) + \underline{\gamma}_{j+1}(\underline{u}_{j+1} - \underline{c}_{j+1})r_1(\underline{w}_{j+1}) = 0$$

for $j = j_0, j_0+1$. The linear independence of r_1 and r_2 again tells us that $\underline{\gamma}_j = \underline{\gamma}_{j+1} = 0$ which means $w_{j-1/2} = w_{j-1}$, $w_{j+1} = w_{j+1/2}$ for $j = j_0, j_0+1$. Thus $w_{j_0} = w_{j_0+1/2} = w_{j_0+1}$.

As a consequence of this we have:

Corollary (4.1)

Any steady discrete shock solution is eventually constant - i.e. there are indices J_1, J_2 such that $w_j \equiv w^L$ if $j < J_1$, $w_j \equiv w^R$ if $j > J_2$.

Now we consider the one shock case. Let j_0 be the largest index so that $c_j < u_j$ for all $j < j_0$. This means that $w_j \equiv w^L$ for $j < j_0-1$, $c_{j_0} < u_{j_0}$ and $c_{j_0+1} > u_{j_0+1}$.

This also means that $\int_{\Gamma^{j_0}} (\partial f(w))^- dw = 0$. Now if $u_{j_0-1/2} < c_{j_0-1/2}$, this integral is a nontrivial linear combination of two linearly independent two vectors (by a now familiar application of the mean value theorem). Thus $u_{j_0-1/2} > c_{j_0-1/2}$.

This implies that equation (4.52) for $j = j_0$ becomes

(4.55) $$f(w_{j_0}) - f(w^L) + \int_{\Gamma^{j_0+1}} (\partial f(w))^- dw = 0 .$$

We wish to show that $u_{j_0+1/2} > c_{j_0+1/2}$. Suppose that this is false. We use our special hypothesis for the first time - $u_{j_0+1} > -c_{j_0+1}$, $u_{j_0+3/2} > c_{j_0+3/2}$. Then equation (4.52), for $j = j_0 + 1$, becomes

(4.56) $$\int_{\Gamma_2^{j_0+1}} (\partial f(w))^+ dw + \int_{\Gamma_1^{j_0+1}} (\partial f(w))^- dw = 0 .$$

Our usual "linear independence" argument, together with the assumption $u_{j_0+1/2} > -c_{j_0+1/2}$, tells us that $w_{j_0+1/2} = w_{j_0}$. This is a contradiction since we assumed $u_{j_0+1/2} < c_{j_0+1/2}$ and $u_{j_0} > c_{j_0}$.

Thus $u_{j_0+1/2} > c_{j_0+1/2}$. This means that we may write (4.55) as

$$f(w_{j_0+1}) + f(w_{j_0}) - f(\bar{\bar{w}}_{j_0+1/2}) - f(w^R) = 0 \ , \tag{4.57}$$

which is exactly (4.27).

Summing equation (4.52) from j to ∞ gives us

$$f(w_j) - f(w^R) = -\int_{\Gamma^{j+1}} (\partial f(w))^- dw = \int_{\Gamma^j} (\partial f(w))^+ dw \ . \tag{4.58}$$

Let $j = j_0+2$ in (4.58) and use the hypothesis that $u_{j_0+3/2} < c_{j_0+3/2}$, $u_{j_0+2} < c_{j_0+2}$, $u_{j_0+3/2} > -c_{j_0+3/2}$, $u_{j_0+3} > -c_{j_0+3}$. We then have:

$$f(w_{j_0+2}) - f(w^R) = -\int_{\Gamma_2^{j_0+3}} (\partial f(w))^- dw = \int_{\Gamma_1^{j_0+2}} (\partial f(w)) \overset{+}{dw} \ . \tag{4.59}$$

By our usual "linear independence" argument, all three terms above vanish, which means $w_{j_0+2} = w^R$, (since this is the only subsonic root of $f(w_{j_0+2}) = f(w^R)$), $w_{j_0+1} = w_{j_0+3/2}$, and $\int_{\Gamma^{j_0+3}} (\partial f(w))^- dw = 0$. Thus w_{j_0+1} is connected to $w_{j_0+2} = w^R$ via a one wave, which means $u_{j_0+1} + \frac{2}{\gamma-1} c_{j_0+1} = u^R + \frac{2}{\gamma-1} c^R$.

Repeating this argument gives us that $w_j \equiv w^R$, $j > j_0+2$.

We must now merely demonstrate that the only solutions of (4.57) (i.e. (4.27)) subject to (4.29) is the family described in the statement of this theorem:

Let $(c_{j_0}, c_{j_0+1}) = (\alpha,\beta)$ be such a solution of (4.57). Then find the solution of the system of ordinary differential equations, described in (a) of the "existence" part of the theorem, which passes through $c_{j_0}, c_{j_0+1})$. The analysis following (4.36) can be easily extended to show that this family of solutions $c_{j_0}(\alpha), c_{j_0+1}(\alpha))$ leaves the rectangle $\bar{c}_{j_0+1/2} < c_{j_0} < c^L$, $c^R < c_{j_0+1} < \bar{\bar{c}}_{j_0+1/2}$ through the $(\bar{\bar{c}}_{j_0+1/2}, c^R)$ and $(c^L, \bar{\bar{c}}_{j_0+1/2})$.

5. ALGORITHMS AND NUMERICAL EXAMPLES

In this section we shall present results from numerical tests with the one dimensional potential flow scheme developed in Section 3. We shall, however, first briefly discuss some other related algorithms in one and two space dimensions. A forthcoming computational paper will contain experiments with some of these algorithms.

The one dimensional algorithm which is presented in this paper can be used as a basic component in an ADI-scheme for the two dimensional full potential equation. The appropriate eigenvectors can be derived analytically and the corresponding upwind scheme for the three components can be constructed. Experiments with this type of upwind schemes for the Euler equations are given in [17].

All methods in this paper and in [17] are explicit. Implicit methods often have the advantage of being stable for longer time steps (Δt). The linearized problem may be unconditionally stable. This is a definite advantage for calculations to steady state and also for some transient problems.

An important part of the computational work when using an implicit method lies in the solution of linear system of equations. We see two ways to reduce the bandwidth of these systems and hence also reduce the work.

The discrete scheme for the potential flow equations can be rewritten in scalar form, compare [8]. There is however no simple closed form for the corresponding scalar difference scheme and approximations are needed. Another possibility is splitting the differencing and the solution of linear systems into two parts; one corresponding to forward and one to backward differencing. The linear systems to be solved are then block bidiagonal and the computational complexity is reduced compared to a straight forward use of Crank-Nicolson or fully implicity schemes.

Our numerical example will be the potential flow equations in one space variable (1.6)

(5.1) $$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \frac{1}{2} u^2 + \frac{A\gamma}{\gamma-1} \rho^{\gamma-1} \end{pmatrix}_x = 0 , \quad 0 < x < 1 , \quad t > 0$$

with initial and boundary conditions

$$\rho(x,0) = \rho(x) \ , \quad u(x,0) = u(x) \tag{5.2}$$

$$\begin{aligned} &\rho(0,t) = \rho_L(t) \ , \quad u(0,t) = U_L(t) \\ &u(1,t) = u_R(t) \end{aligned} \tag{5.3}$$

The boundary conditions (5.3) in our example correspond to supersonic inflow at $x = 0$ and subsonic outflow at $x = 1$. Other combinations have also been studied and they give the same qualitative behavior except for the fully supersonic case. When $|u| > c$ the scheme derived in Section 3 will reduce to the standard upwind scheme without switches. Shocks will then not have sharp profiles as was also the case in scalar approximations [5].

The following values we have chosen agree with those in the examples in [8].

$$\gamma = 1.4$$

$$A = 1/(1.4 \cdot 1.2^2) \approx 0.5$$

$$\rho_L(t) \equiv u_L(t) \equiv 1 \ .$$

The value of u on the right will be changed between the two examples presented here.

Example 1: This is exactly the problem considered in [8] with $u_R(t) = 0.8$. The one shock solution we will approximate, is characterized by the right state $\rho_R(t) = 1.265$ and the shock velocity $V_S = 0.0447$. In the scheme (2.9) the above value of $\rho_R(t)$ was given as numerical boundary condition. See Section 3 for the influence of numerical boundary conditions. See Figure 5.1 and 5.2 for computational results and parameters. The shock is essentially resolved over two mesh points even if the theory of Section 4 does not cover this case.

Example 2: In this test we modify the outflow values in order to achieve a steady shock $u_R(t) = 0.728$, $\rho_R(t) = 1.373$. Figures 5.3 and 5.4 display the results of the computation. The shock is resolved exactly over two mesh points and the convergence history does not produce any overshoots.

$$\rho(x,0) = \rho(x) \ , \quad u(x,0) = u(x) \tag{5.2}$$

$$\begin{aligned} &\rho(0,t) = \rho_L(t) \ , \quad u(0,t) = U_L(t) \\ &u(1,t) = u_R(t) \end{aligned} \tag{5.3}$$

The boundary conditions (5.3) in our example correspond to supersonic inflow at $x = 0$ and subsonic outflow at $x = 1$. Other combinations have also been studied and they give the same qualitative behavior except for the fully supersonic case. When $|u| > c$ the scheme derived in Section 3 will reduce to the standard upwind scheme without switches. Shocks will then not have sharp profiles as was also the case in scalar approximations [5].

The following values we have chosen agree with those in the examples in [8].

$$\gamma = 1.4$$

$$A = 1/(1.4 \cdot 1.2^2) = 0.5$$

$$\rho_L(t) \equiv u_L(t) \equiv 1 \ .$$

The value of u on the right will be changed between the two examples presented here.

<u>Example 1:</u> This is exactly the problem considered in [8] with $u_R(t) = 0.8$. The one shock solution we will approximate, is characterized by the right state $\rho_R(t) = 1.265$ and the shock velocity $V_S = 0.0447$. In the scheme (2.9) the above value of $\rho_R(t)$ was given as numerical boundary condition. See Section 3 for the influence of numerical boundary conditions. See Figure 5.1 and 5.2 for computational results and parameters. The shock is essentially resolved over two mesh points even if the theory of Section 4 does not cover this case.

<u>Example 2:</u> In this test we modify the outflow values in order to achieve a steady shock $u_R(t) = 0.728$, $\rho_R(t) = 1.373$. Figures 5.3 and 5.4 display the results of the computation. The shock is resolved exactly over two mesh points and the convergence history does not produce any overshoots.

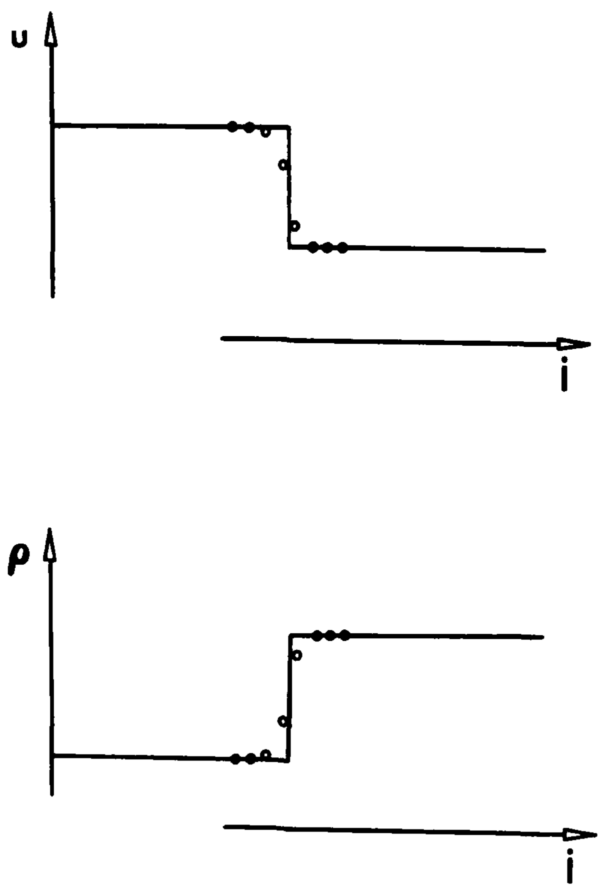
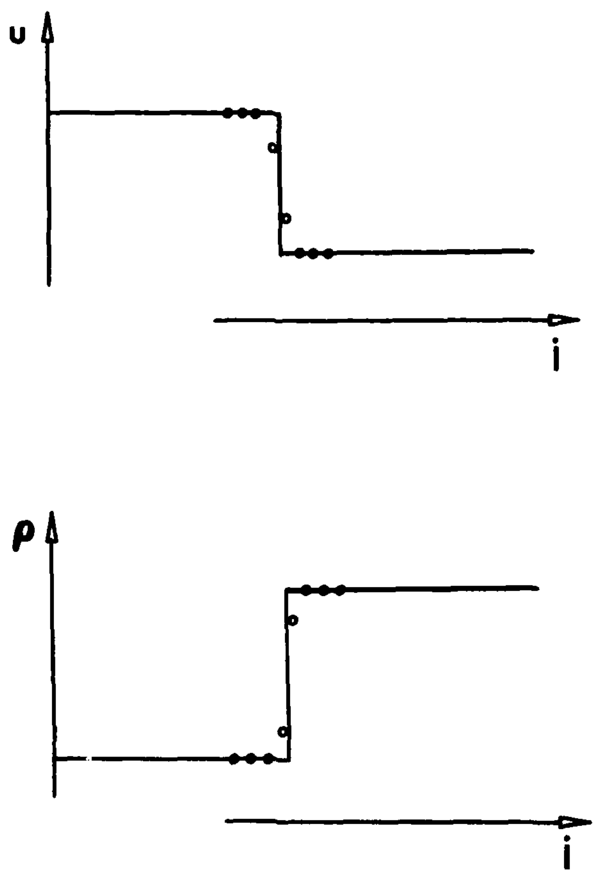


Figure 5.1, 5.2: Shock profiles for example 1, section 5. Results after 50 iterations. Exact initial values. Courant number 0.75.



<u>Figure</u> 5.3, 5.4: Shock profiles for example 2, section 5. Results after 150 iterations. Piecewise linear initial values. Courant number o.75.

References

1. Ballhaus, W. F. and P. M. Goorjian, Implicit finite difference computations of unsteady transonic flow about airfoils, including the treatment of irregular shock wave motions, AIAA report no. 77-205, 1977.

2. Chipman, R. and A. Jameson, Fully conservative numerical solutions for unsteady irrotational flow about airfoils, AIAA report no. 79 - 1555, 1979.

3. Cole, J. D., Modern developments in transonic flow, SIAM J. Appl. Math. v. 29, 1975, pp. 763-786.

4. Crandall, M. G. and A. Majda, Monotone difference approximations for scalar conservation laws. Math. Comp. v. 34, 1980, pp. 1-22.

5. Engquist, B. and S. Osher, Stable and entropy satisfying approximations for transonic flow calculations, Math. Comp., v. 34, 1980, pp. 45-75.

6. Engquist, B. and S. Osher, One sided difference approximations for nonlinear conservation laws, to appear in Math. Comp., 1981.

7. Engquist, B. and S. Osher, One sided difference schemes and transonic flow, Proc. Nat. Acad. Sci., v. 77, 1980, pp. 3071-3074.

8. Goorjian, P. M., Implicit computations of unsteady transonic flow governed by the conservative full potential equation in conservation form. AIAA report no. 80-0150, 1980.

9. Harten, A., J. M. Hyman and P. Lax, On finite difference approximations and entropy conditions for shocks, Comm. Pure. Appl. Math., v. 29, 1976, pp. 297-332.

10. Jameson, A., Numerical solutions of nonlinear partial differential equations of mixed type, Numerical Solutions of Partial Differential Equations III. Academic Press, New York 1976, pp. 275-320.

11. Jennings, G., Discrete shocks, Comm. Pure. Appl. Math., v. 27, 1974, pp. 25-37.

12. Krupp, J. A. and J. D. Cole, Studies in transonic flow IV, unsteady transonic flow, UCLA Eng. Dept. report no. 76104, 1976.

13. Lax, P. D., Shock waves and entropy, Contributions to Nonlinear Functional Analysis, Academic Press, New York, 1971, pp. 603-634.

14. Lax, P. D. and B. Wendroff, Systems of conservation laws. Comm. Pure. Appl. Math., v. 23, 1960, pp. 217-237.

15. Mock, M. S., Some higher order difference equations enforcing an entropy inequality, Mich. Math. J., 1978, pp. 325-344.

16. Murman, E. M. and J. D. Cole, Calculations of plane steady transonic flows, AIAA J. v. 9, 1971, pp. 114-121.

17. Osher, S. and Solomon, Upwind difference schemes for compressible gas dynamics, submitted to Math. Comp.

18. Van Leer, B. Towards the ultimate conservative difference scheme IV; A new approach to numerical convection, J. Comp. Phys., v. 23, 1977, pp. 276-299.

19. Warming, R. F. and R. M. Beam, Upwind second order difference schemes and applications in aerodynamical flows, AIAA J., v. 14, 1976, pp. 1241-1249.

20. Warming, R. F. and R. M. Beam, On the construction and application of implicit factored schemes for conservation laws, Computational Fluid Dynamics, SIAM-AMS Proceedings, v. 11, 1978, pp. 85-129.

21. Roache, P., Computational fluid dynamics, Hermosa Publishers, 1972.

BE:SO/db

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 2186	2. GOVT ACCESSION NO.: AD-A099 348	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): UPWIND DIFFERENCE SCHEMES FOR SYSTEMS OF CONSERVATION LAWS - POTENTIAL FLOW EQUATIONS		5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Bjorn Engquist and Stanley Osher		8. CONTRACT OR GRANT NUMBER(s): DAAG29-80-C-0041; NCA2-OR390-002
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 3- Numerical Analysis and Computer Science
11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below		12. REPORT DATE: March 1981
		13. NUMBER OF PAGES: 46
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report): UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

NASA-AMES University Consortium
Los Angeles, California 90024

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Finite difference approximations, upwind schemes, potential flow equations, entropy condition, discrete shocks

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We derive new upwind type finite difference approximations to systems of nonlinear hyperbolic conservation laws. The general technique is exemplified by the potential flow equations written as a first order system. The scheme has desirable properties for shock calculations. For the potential flow approximation, we show that the entropy condition is valid for limit solutions and that there exist discrete steady shocks which are unique and sharp. Numerical examples are given.

DATE
ILME